SUPERBOSONISATION VIA RIESZ SUPERDISTRIBUTIONS

ALEXANDER ALLDRIDGE AND ZAIN SHAIKH

Abstract. The superbosonisation identity of Littelmann–Sommers–Zirnbauer is a new tool to study universality of random matrix ensembles *via* supersymmetry, which is applicable to non-Gaussian invariant distributions. In this note, we identify the right-hand side with a super-generalisation of the Riesz distribution. Using the Laplace transformation and tools from harmonic superanalysis, we give a short and conceptual new proof of the formula.

1. Introduction

Supersymmetry was introduced in physics as a means to formulate Bose–Fermi symmetry in quantum field theory. Since the advent of supergravity, it is usually connected to superstring theory. Although this relationship is indeed intimate and fundamental, supersymmetry is also deeply rooted in the physics of condensed matter. The so-called supersymmetry method [7] has been used to great effect in the study of disordered systems, in particular in connection to the metal-insulator transition, in other words, in the analysis of localisation and delocalisation phenomena for certain random matrix ensembles.

In its traditional form, based on the so-called Hubbard–Stratonovich transformation, the method is applicable chiefly to normally distributed ensembles. To extend its range beyond Gaussian disorder, for instance to establish universality for invariant ensembles, a complementary tool, dubbed 'superbosonisation', was introduced by Littelmann–Sommers–Zirnbauer in their seminal paper [20].

In general, the superbosonisation identity holds in the unitary, orthogonal, and unitary-symplectic symmetry cases. We consider only unitary symmetry here, although our methods and results are not restricted to this case.

One considers the space $W = \mathbb{C}^{p|q\times p|q}$ of square super-matrices and a certain subsupermanifold Ω of purely even codimension with underlying manifold

$$\Omega_0 = \mathrm{Herm}^+(p) \times \mathrm{U}(q),$$

the product of the positive Hermitian $p\times p$ matrices with the unitary $q\times q$ matrices.

Let f be a superfunction defined and holomorphic on the tube domain based on $\mathrm{Herm}^+(p) \times \mathrm{Herm}(q)$. If f has sufficient decay at infinity along Ω_0, then the superbosonisation identity states

$$\int_{\mathbb{C}^{p|q\times n}} |Dv|\, f(Q(v)) = C \int_{\Omega} |Dy|\, \mathrm{Ber}(y)^n f(y), \tag{1.1}$$

where C is some finite positive constant. Here, Q is the quadratic map $Q(v) = vv^*$, $|Dv|$ is the standard Berezinian density, and $|Dy|$ is a Berezinian density on Ω,

2010 *Mathematics Subject Classification.* Primary 22E30, 58C50; Secondary 22E45, 46S60.

Key words and phrases. gamma function, Laplace transform, Lie supergroup, Riesz distribution, superbosonisation, supersymmetry method, symmetric space.

This research was funded by the Leibniz junior indepenent research group grant and the SFB/TR 12 grant "Symmetries and Universality in Mesoscopic Systems", provided by Deutsche Forschungsgemeinschaft (DFG).

invariant under a natural transitive supergroup action. The precise meaning of all the quantities involved will be made clear in the course of the paper.

A remarkable special case of the identity occurs when $p = 0$. Then Equation (1.1) reduces to

$$\int_{\mathbb{C}^{0|q\times n}} |Dv|\, f(Q(v)) = C \int_{\mathrm{U}(q)} |Dy|\, \det(y)^{-n} f(y),$$

which is known as the bosonisation identity in physics. Notice that the left-hand side is a purely fermionic Berezin integral, whereas the right-hand side is purely bosonic. Formally, it turns fermions $\psi\bar{\psi}$ into bosons $e^{i\varphi}$.

If in addition $q = 1$, then the left-hand side is $f^{(n)}(0)$ up to some constant, and we obtain the Cauchy integral formula.

On the other hand, if $q = 0$, then $\Omega = \mathrm{Herm}^+(p)$, and the right-hand side

$$\langle T_n, f\rangle := \int_{\mathrm{Herm}^+(p)} |Dy|\, \det(y)^n f(y)$$

is the so-called (unweighted) *Riesz distribution*. In this case, Equation (1.1) is due to Ingham and Siegel [15, 33]. Moreover, it is known to admit a far-reaching generalisation in the framework of Euclidean Jordan algebras [9].

This observation links the identity to equivariant geometry, Lie theory, and harmonic analysis; and thus forms the starting point of the present paper. Our strategy is to exploit the transitive action of a certain supergroup on Ω to compute the Laplace transform $\mathscr{L}(T_n)$ of the functionals T_n. The corresponding transformed identity is easy to verify, since the Laplace transform of the left-hand side is straightforward to evaluate.

Of course, since the geometry is more complicated for $q > 0$, the evaluation of the right-hand side $\mathscr{L}(T_n)$ becomes more delicate. Also, a theory of Laplace transforms for superdistributions had to be developed, because it was unavailable in the literature. Here, a technical difficulty is that T_n is not obviously a superdistribution (although *a posteriori*, Equation (1.1) shows that it is), but rather a functional on a space of holomorphic superfunctions. Thus, we are obliged to study the Laplace transformation on various different spaces of generalised functions.

Besides providing a conceptual framework in which Equation (1.1) follows with relative ease, our approach also establishes a connection to analytic representation theory that previously went unnoticed. Namely, for a suitable choice of normalisation, the constant C in Equation (1.1) is

$$C = \sqrt{\pi}^{np}\, \Gamma_\Omega(n)^{-1},$$

where $\Gamma_\Omega(n)$ is the evaluation at $(n, \dots, n)$ of a the meromorphic function of $p + q$ indeterminates, known as the Gindikin Γ function for $q = 0$. In Theorem 3.19, we explicitly determine $\Gamma_\Omega(\mathbf{m})$ for any $\mathbf{m} = (m_1, \dots, m_{p+q})$, as follows:

$$(2\pi)^{p(p-1)/2} \prod_{j=1}^{p} \Gamma(m_j - (j-1)) \prod_{k=1}^{q} \frac{\Gamma(q-(k-1))}{\Gamma(m_{p+k} + q - (k-1))} \frac{\Gamma(m_{p+k} + k)}{\Gamma(m_{p+k} - p + k)}.$$

Note that this function has zeros for $q > 0$, whereas for $q = 0$, it only has poles.

When $q = 0$, this function is closely related to the c-function of the Riemannian symmetric space Ω. Moreover, the renormalised Riesz distribution $R_n := \Gamma_\Omega(n)^{-1} T_n$ defines the unitary structure on the holomorphic discrete series representation of $\mathrm{U}(p,p)$ whose lowest $\mathrm{U}(p)$-type is the character $\det(z)^n$ [8, 28].

The Gindikin Γ function Γ_Ω also appears in the b-function equation for the relatively invariant polynomial $\det(z)$, *via*

$$\det\big(\tfrac{\partial}{\partial z}\big) \det(z)^n = n(n+1)\cdots(n+p-1) \det(z)^{n-1} = (-1)^p \frac{\Gamma_\Omega(1-n)}{\Gamma_\Omega(-n)} \det(z)^{n-1}.$$

This leads to a functional equation for R_n which can be exploited to give a meromorphic extension of R_n as a distribution-valued function of n. This fact was famously applied by Rossi and Vergne in their proof of the analytic extension of the holomorphic discrete series [28]. The implications of these connections for the representation theory of supergroups will be investigated in a forthcoming paper.

Let us give a brief synopsis of the paper's contents. In Section 2, we give the basic setup for the statement and proof of the superbosonisation formula, introducing the relevant supergroups and the functionals which define both sides of the equation. We give a proof of the identity in Theorem 2.6, up to the computation of the Laplace transform $\mathscr{L}(T_n)$ of the right-hand side, which is deferred to Section 3. In that section, we actually discuss more generally the so-called conical superfunctions attached to Ω, and determine their Laplace transforms. The main step is the explicit determination of the Gindikin Γ function Γ_Ω, in Theorem 3.19.

The paper is complemented with an extensive Appendix. In Appendix A.1, we discuss the language of (generalised) points, which will be an indispensable tool. Appendix B covers the theory of Berezinian fibre integrals, which is used throughout Section 3. Finally, the theory of generalised superfunctions and their Laplace transforms is developed in Appendix C. These techniques form the basis of our proof of the superbosonisation identity in Theorem 2.6.

Acknowledgement. We wish thank Martin Zirnbauer for extensive discussions and for bringing this topic to our attention. The first named author wishes to thank Jacques Faraut for his interest, and the second named author wishes to thank Bent Ørsted for some useful comments.

2. The superbosonisation identity

In this section, we set up the basic framework to formulate the superbosonisation identity. We reduce the proof to a super version of the theory of the Laplace transform of generalised functions, which is discussed in Appendix C, and the explicit computation of certain Laplace transforms, which is performed in Section 3.

2.1. **Preliminaries.** The machinery of supergeometry will be used freely. The reader may, for instance, consult Refs. [4,10,19,22]. Here, we briefly fix our notation and highlight points in which we deviate slightly from the standard lore. More subtle matters, concerning the functor of points and the theory of super-integration, are summarised in Appendices A and B.

We will work exclusively in the category of $\mathbb{C}$-superspaces and certain full subcategories thereof. By definition, a $\mathbb{C}$-superspace is a pair $X = (X_0, \mathcal{O}_X)$ where X_0 is a topological space and $\mathcal{O}_X$ is a sheaf of unital supercommutative superalgebras over $\mathbb{C}$, whose stalks $\mathcal{O}_{X,x}$ are local rings. A morphism $f : X \to Y$ is a pair $(f_0, f^\sharp)$ comprising a continuous map $f_0 : X_0 \to Y_0$ and a sheaf map $f^\sharp : f_0^{-1}\mathcal{O}_Y \to O_X$, which is local in the sense that $f^\sharp(\mathfrak{m}_{Y,f_0(x)}) \subseteq \mathfrak{m}_{X,x}$ for any x. Global sections $f \in \Gamma(\mathcal{O}_X)$ of $\mathcal{O}_X$ are called *superfunctions*. Due to the locality condition, the *value* $f(x) := f + \mathfrak{m}_{X,x} \in \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = \mathbb{C}$ is defined for any x. *Open subspaces* of a $\mathbb{C}$-superspace X are given by $(U, \mathcal{O}_X|_U)$, for any open subset $U \subseteq X_0$.

We consider two types of model spaces. Firstly, when $V = V_{\bar 0} \oplus V_{\bar 1}$ is a complex super-vector space, then we define $\mathcal{O}_V := \mathscr{H}_{V_{\bar 0}} \otimes \bigwedge(V_{\bar 1})^*$ where $\mathscr{H}$ denotes the sheaf of holomorphic functions. The space $L(V) := (V_{\bar 0}, \mathcal{O}_V)$ is called the *complex affine superspace* associated with V. Secondly, if in addition, we are given a real form $V_{\bar 0,\mathbb{R}}$ of $V_{\bar 0}$, then the pair $(V, V_{\bar 0,\mathbb{R}})$ is called a cs *vector space*, and we define $\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}} := \mathcal{C}^\infty_{V_{\bar 0,\mathbb{R}}} \otimes \bigwedge(V_{\bar 1})^*$, where $\mathcal{C}^\infty$ denotes the sheaf of complex-valued smooth

functions. The space $L(V,V_{\bar 0,\mathbb{R}}) \coloneqq (V_{\bar 0,\mathbb{R}},\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}})$ is called the cs *affine superspace* associated with $(V,V_{\bar 0,\mathbb{R}})$. (The *cs* terminology is due to J. Bernstein.)

Consider now a superspace X whose underlying topological space X_0 is Hausdorff and which admits a cover by open subspaces which are isomorphic to open subspaces of some $L(V)$ (resp. $L(V,V_{\bar 0,\mathbb{R}})$), where V (resp. $(V,V_{\bar 0,\mathbb{R}})$) may vary. Then X is called a *complex supermanifold* (resp. a cs *manifold*). Both complex supermanifolds and *cs* manifolds form full subcategories of the category of $\mathbb{C}$-superspaces, which admit finite products. Moreover, the assignment which sends $L(V)$ to $L(W,W_{\bar 0,\mathbb{R}})$, where $W_{\bar 0} \coloneqq V_{\bar 0}\otimes_{\mathbb{R}}\mathbb{C}$, $W_{\bar 1} \coloneqq V_{\bar 1}$, and $W_{\bar 0,\mathbb{R}} \coloneqq V_{\bar 0}$, extends to a product-preserving functor from complex supermanifolds to *cs* manifolds. This functor is called *cs*-ification; the *cs*-ification of X is denoted by X_{cs}.[1]

Group objects in the category of complex supermanifolds (resp. *cs* manifolds) are called complex Lie supergroups (resp. *cs* Lie supergroups). The category of complex Lie supergroups is equivalent to the category of complex supergroup pairs, *cf.* [10]. These are pairs $(\mathfrak{g},G_0)$ consisting of a complex Lie superalgebra $\mathfrak{g}$ and complex Lie group G_0, such that $\mathfrak{g}_{\bar 0}$ is the Lie algebra of G_0, endowed with an action Ad of G_0 on $\mathfrak{g}$ by Lie superalgebra automorphisms, which extends the adjoint action of G_0 on $\mathfrak{g}_{\bar 0}$, and whose derivative coincides with the restriction of the bracket of $\mathfrak{g}$. Morphisms $(\mathfrak{g},G_0)\to(\mathfrak{h},H_0)$ of such pairs are given by pairs $(d\phi,\phi_0)$ of a morphism ϕ_0 of Lie groups and a ϕ_0-equivariant morphism $d\phi$ of Lie superalgebras. By assuming instead that G_0 be a real Lie group and that $\mathfrak{g}_{\bar 0}$ is the complexification of the Lie algebra of G_0, and modifying all definitions accordingly, one obtains the category of *cs* supergroup pairs, which is equivalent to the category of *cs* Lie supergroups.

Given a Lie supergroup (complex or *cs*), a *closed subsupergroup* is a closed subsupermanifold which is a Lie supergroup, such that the embedding morphism is a morphism of supergroups. In particular, given a complex Lie supergroup G and a closed *cs* subsupergroup H of G_{cs}, then we say that H is a cs *form* of G if the Lie superalgebra $\mathfrak{g}$ of G coincides with that of H, and H_0 is a real form of G_0, or equivalently, that $(\mathfrak{g},H_0)$ is a *cs* supergroup pair. In view of the above remarks, to define a *cs* form of G, it suffices to specify an embedded real form H_0 of G_0.

2.2. **The relevant supergroups.** We now begin building the natural framework for the superbosonisation identity proper. Its right-hand side is a certain equivariant integral over a symmetric superspace, and we now introduce the supergroups which are relevant to its precise definition. Recall facts and definition from Appendix A.

Consider the complex Lie supergroup $G'_{\mathbb{C}} \coloneqq \mathrm{GL}(2p|2q,\mathbb{C})$, with Lie superalgebra $\mathfrak{g}' = \mathfrak{gl}(2p|2q,\mathbb{C})$.[2] The underlying Lie group of $G'_{\mathbb{C}}$ is $G'_{\mathbb{C},0} = \mathrm{GL}(2p,\mathbb{C})\times\mathrm{GL}(2q,\mathbb{C})$. We will write S-valued points $g\in_S G'_{\mathbb{C}}$ in the non-standard form

$$\begin{array}{cc} & \begin{array}{cc} p|q & p|q \end{array} \\ \begin{array}{c} p|q \\ p|q \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

rather than in the more customary even-odd decomposition. Although this may seem unnatural at first, it will bear fruit in the later sections of this paper.

---

[1]In passing, note that E. Witten has recently [35] advocated the study of *cs* submanifolds of complex supermanifolds; in this setting, the *cs*-ification is the universal such *cs* submanifold.

[2]The notation comes from the fact that $G'_{\mathbb{C}}$ arises as the Howe dual partner of the Lie group $G_{\mathbb{C}} \coloneqq \mathrm{GL}(n,\mathbb{C})$ in the oscillator representation of $\mathfrak{spo}(V)$, where $V \coloneqq \mathbb{C}^{p|q\times n}\oplus\mathbb{C}^{n\times p|q}$.

We define a Lie supergroup $K_{\mathbb{C}} := \mathrm{GL}(p|q,\mathbb{C}) \times \mathrm{GL}(p|q,\mathbb{C})$, which is a closed complex subsupergroup of $G'_{\mathbb{C}}$, embedded as

$$\begin{array}{c c} & \begin{array}{cc} p|q & p|q \end{array} \\ \begin{array}{c} p|q \\ p|q \end{array} & \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \end{array},$$

with $A, D \in_S \mathrm{GL}(p|q,\mathbb{C})$. Further, $\mathfrak{k} := \mathrm{Lie}(K_{\mathbb{C}})$.

We now define two *cs* forms K and H of $K_{\mathbb{C}}$ that are of interest. We do so by specifying real forms of $K_{\mathbb{C},0}$ by

$$K_0 := \mathrm{U}(p) \times \mathrm{U}(q) \times \mathrm{U}(p) \times \mathrm{U}(q)$$

and

$$H_0 := \mathrm{GL}(p,\mathbb{C}) \times \mathrm{U}(q) \times \mathrm{U}(q).$$

Here, the latter is embedded into $K_{\mathbb{C},0}$ as the set of all matrices of the form

$$\begin{array}{c c} & \begin{array}{cccc} p & q & p & q \end{array} \\ \begin{array}{c} p \\ q \\ p \\ q \end{array} & \begin{pmatrix} A & & & \\ & D & & \\ & & (A^*)^{-1} & \\ & & & D' \end{pmatrix} \end{array}$$

with $A \in \mathrm{GL}(p,\mathbb{C})$, $D, D' \in \mathrm{U}(q)$.

To conclude this section, we define the homogeneous superspace $\Omega := H/H \cap K$. It is a *cs* manifold, whose underlying manifold is the homogeneous space

$$\Omega_0 = \mathrm{Herm}^+(p) \times \mathrm{U}(q),$$

where $\mathrm{Herm}^+(p)$ is the cone of positive definite Hermitian $p \times p$ matrices.

2.3. **The Q-morphism.** The superbosonisation identity describes the transformation of an integral under a certain quadratic morphism Q. We introduce it in a form convenient for our purposes.

Consider the super-vector spaces

$$V := \mathbb{C}^{p|q\times n} \oplus \mathbb{C}^{n\times p|q} \quad \text{and} \quad W := \mathbb{C}^{p|q}.$$

We define a partial action of $G'_{\mathbb{C}}$ on $L(W)$. Namely, for

$$g' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in_S G'_{\mathbb{C}}$$

and $Z \in_S W$, the action is by fractional linear transformations

$$g'.Z = (AZ + B)(CZ + D)^{-1},$$

whenever $CZ + D \in_S \mathrm{GL}(p|q,\mathbb{C})$. Observe that if $g \in_S K_{\mathbb{C}}$, this condition is always satisfied, so that $K_{\mathbb{C}}$ acts on $L(V)$. This induces actions of the *cs* forms K and H of $K_{\mathbb{C}}$. Notice that $\Omega = H.1$, since the isotropy supergroup of H at $1 \in W$ is $H \cap K$.

Now, consider the supervector space $U := \mathbb{C}^{(n+p|q)\times(n+p|q)}$. We have an embedding $V \hookrightarrow U$, given by

$$\begin{array}{c c} & \begin{array}{cc} n & p|q \end{array} \\ \begin{array}{c} n \\ p|q \end{array} & \begin{pmatrix} 0 & a' \\ a & 0 \end{pmatrix} \end{array},$$

where $(a, a') \in V$. Further, we have an embedding of W into U, given by

$$\begin{array}{c c} & \begin{array}{cc} n & p|q \end{array} \\ \begin{array}{c} n \\ p|q \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix} \end{array},$$

where $w \in W$. We obtain embeddings $L(V) \to L(U)$ and $L(W) \to L(U)$ as closed complex subsupermanifolds.

A real form of $U_{\bar 0}$ is given by $U_{\bar 0,\mathbb{R}} = \mathrm{Herm}(n+p) \times \mathrm{Herm}(q)$, *i.e.* the set of matrices

$$\begin{array}{c|c} & \begin{array}{ccc} n & p & q \end{array} \\ \begin{array}{c} n \\ p \\ q \end{array} & \begin{pmatrix} b_1 & b_2 & 0 \\ b_3 & b_4 & 0 \\ 0 & 0 & b' \end{pmatrix} \end{array}$$

with $\left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) \in \mathrm{Herm}(n+p)$, $b' \in \mathrm{Herm}(q)$. We set

$$V_{\bar 0,\mathbb{R}} \coloneqq V \cap U_{\bar 0,\mathbb{R}} \quad \text{and} \quad W_{\bar 0,\mathbb{R}} \coloneqq W \cap W_{\bar 0,\mathbb{R}}.$$

These are real forms of $V_{\bar 0}$ and $W_{\bar 0}$, respectively. We obtain *cs* manifolds denoted by $L(U, U_{\bar 0,\mathbb{R}})$, $L(V, V_{\bar 0,\mathbb{R}})$, and $L(W, W_{\bar 0,\mathbb{R}})$.

For $x, y \in U$, we define

$$P(x)y \coloneqq xyx \in U.$$

Since this expression is polynomial, there is a unique morphism $L(U) \times L(U) \to L(U)$ given on S-valued points by $(x, y) \mapsto P(x)y$. We let $Q : L(V) \to L(W)$ denote the morphism given by $Q(x) = P(x)c_n$ where

$$c_n = \begin{array}{c|c} & \begin{array}{cc} n & p|q \end{array} \\ \begin{array}{c} n \\ p|q \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}.$$

In the following, recall the definition of the Schwartz space and the space of tempered superdistributions from Appendix C.1.

Proposition 2.1. *Let the* squaring morphism $P : L(U) \to L(U)$ *be defined by* $P(x) \coloneqq P(x)1_{n+p|q}$. *Then* P *induces a morphism* $L(V, V_{\bar 0,\mathbb{R}}) \to L(U, U_{\bar 0,\mathbb{R}})$ *of* cs *manifolds, and the pullback along this morphism induces a continuous linear map* $P^\sharp : \mathscr{S}(U, U_{\bar 0,\mathbb{R}}) \to \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$.

Proof. For a *cs* manifold S, let $x \in_S V$. Then

$$x = \begin{array}{c|c} & \begin{array}{cc} n & p|q \end{array} \\ \begin{array}{c} n \\ p|q \end{array} & \begin{pmatrix} 0 & a' \\ a & 0 \end{pmatrix} \end{array} \quad \text{implies} \quad P(x) = \begin{array}{c|c} & \begin{array}{cc} n & p|q \end{array} \\ \begin{array}{c} n \\ p|q \end{array} & \begin{pmatrix} a'a & 0 \\ 0 & aa' \end{pmatrix} \end{array}.$$

The underlying values of the entries of x satisfy $a_0 = (a'_0)^*$, so $P(x)_0$ is an S_0-point of U_0. Thus, P descends to a morphism of *cs* manifolds $L(V, V_{\bar 0,\mathbb{R}}) \to L(U, U_{\bar 0,\mathbb{R}})$, proving the first statement.

As for the second statement, the partial derivatives of the components of P are polynomials, so it is sufficient to prove that for any $f \in \mathscr{S}(U, U_{\bar 0,\mathbb{R}})$, $k \in \mathbb{N}$,

$$\sup_{x \in V_{\bar 0,\mathbb{R}}} \left| \det(1 + x^2)^k P^\sharp(f)(x) \right| < \infty,$$

and that this quantity, as a function of f, is a continuous seminorm on $\mathscr{S}(U, U_{\bar 0,\mathbb{R}})$.

But for $x \in_S L(V, V_{\bar 0,\mathbb{R}})$, we have

$$\det(1 + x^2) = P^\sharp(\det(1 + z))(x),$$

where $z = \mathrm{id}_U$ denotes the generic point of $L(U, U_{\bar 0,\mathbb{R}})$, and $\det(1+z)$ is a tempered superfunction on $L(U, U_{\bar 0,\mathbb{R}})$, so the statement is immediate. ☐

Lemma 2.2. *The projection morphism* $E : L(U) \to L(W)$ *induces a morphism of* cs *manifolds* $E : L(U, U_{\bar 0,\mathbb{R}}) \to L(W, W_{\bar 0,\mathbb{R}})$*, and the pullback along this morphism gives a continuous linear map* $E^\sharp : \mathscr{S}(W, W_{\bar 0,\mathbb{R}}) \to \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$.

Corollary 2.3. *The morphism $Q : L(U) \to L(U)$ induces a morphism of* cs *manifolds $L(V, V_{\bar{0},\mathbb{R}}) \to L(W, W_{\bar{0},\mathbb{R}})$, and the pullback along this morphism induces a continuous linear map $Q^\sharp : \mathscr{S}(W, W_{\bar{0},\mathbb{R}}) \to \mathscr{S}(V, V_{\bar{0},\mathbb{R}})$.*

Proof. This is immediate, in view of the equation $Q(x) = (E \circ P)(x)$. □

Corollary 2.4. *There is a continuous linear map $Q_\sharp : \mathscr{S}'(V, V_{\bar{0},\mathbb{R}}) \to \mathscr{S}'(W, W_{\bar{0},\mathbb{R}})$, given by*

$$\big\langle Q_\sharp(u), f\big\rangle := \big\langle u, Q^\sharp(f)\big\rangle \quad \text{for all} \quad u \in \mathscr{S}'(V, V_{\bar{0},\mathbb{R}})\,,\ f \in \mathscr{S}(W, W_{\bar{0},\mathbb{R}}).$$

2.4. **Statement of the Theorem.** We can now state our Main Theorem as an identity of two generalised superfunctions and give a proof, up to the explicit computation of the Laplace transform of the right-hand side.

We start by describing both sides of the superbosonisation identity. For this, recall from Appendix B the basics of parameter-dependent super-integration, and the facts and definitions from Appendix C pertaining to generalised superfunctions and their Laplace transforms.

On U and the subspaces V and W, we consider the supertrace form $\mathrm{str}(uu')$. Note that this form is positive definite on $V_{\bar{0},\mathbb{R}}$. Let $|Dv|$ be the standard Berezinian density on $L(V, V_{\bar{0},\mathbb{R}})$ associated with this choice of supersymmetric form (which is well-defined up to some sign). We consider $|Dv| \in \mathscr{S}'(V, V_{\bar{0},\mathbb{R}})$ *via*

$$\langle |Dv|, f\rangle := \int_V |Dv|\, f(v) \quad \text{for all} \quad f \in \mathscr{S}(V, V_{\bar{0},\mathbb{R}}),$$

where the integral is taken with respect to the standard retraction. (Both $|Dv|$ and the standard retraction are defined at the beginning of Appendix C.3.)

Proposition 2.5. *We have $Q_\sharp(|Dv|) \in \mathscr{S}'(W, W_{\bar{0},\mathbb{R}})$ and*

$$\operatorname{supp} Q_\sharp(|Dv|) \subseteq \overline{\mathrm{Herm}^+(p)} \times 0.$$

In particular,

$$\gamma^\circ_{\mathscr{S}}\big(Q_\sharp(|Dv|)\big)_0 \supseteq \mathrm{Herm}^+(p) \times \mathrm{Herm}(q).$$

Proof. The first statement follows from the definition of Q. Since the cone $\mathrm{Herm}^+(p)$ is self-dual, the second statement is immediate from Corollary C.38. □

We call the tempered superdistribution $Q_\sharp(|Dv|)$ the *left-hand side* of the superbosonisation identity.

The homogeneous *cs* manifold $\Omega = H/H \cap K$ is a locally closed *cs* submanifold of $L(W)_{cs}$. It admits a non-zero H-invariant Berezinian density $|Dy|$, which we construct explicitly in Subsection 3.2 below.

When $n \geqslant p$, define $T_n \in \mathscr{Z}'(W, W_{\bar{0},\mathbb{R}})$ by

$$\langle T_n, f\rangle := \int_\Omega |Dy|\, \mathrm{Ber}(y)^n f(y)$$

for any $f \in \mathscr{Z}(W, W_{\bar{0},\mathbb{R}})$. Then T_n is called the *right-hand side* of the superbosonisation identity. For $q = 0$, it coincides with the unweighted *Riesz distribution* for the parameter n, *v.* Ref. [9]. Thus, it may also be called the *Riesz superdistribution.* (That it actually is a superdistribution follows from Theorem 2.6 below.)

Without referring to the theorem, we see that T_n extends as continuous functional to the space of all (holomorphic) superfunctions f on the open subspace of $L(W)$ whose underlying set is the tube

$$T(\gamma_0) := \big(\mathrm{Herm}^+(p) \times \mathrm{Herm}(q)\big) + iW_{\bar{0},\mathbb{R}} = T(\mathrm{Herm}^+(p)) \times \mathbb{C}^{q\times q}, \tag{2.1}$$

and which satisfy Paley–Wiener type estimates along $T(\mathrm{Herm}^+(p))$, *i.e.*

$$\sup\nolimits_{z\in T(\mathrm{Herm}^+(p))}\big|e^{-R\|\Im z\|}(1 + \|z\|)^N f(D; z, w)\big| < \infty \tag{2.2}$$

for any $D \in S(W)$, $N \in \mathbb{N}$, $w \in \mathbb{C}^{q\times q}$, and some $R > 0$. This extension is given by the same formula. All of this follows from Proposition 3.14 below.

We now state the functional analytic version of the superbosonisation identity.

Theorem 2.6 (Superbosonisation identity)**.** *Assume that $n \geqslant p$. Then we have that $T_n \in \mathscr{S}'(W, W_{\bar{0},\mathbb{R}})$, and*

$$Q_\sharp(|Dv|) = \frac{\sqrt{\pi}^{np}}{\Gamma_\Omega(n\mathbb{1})} \cdot T_n,$$

where the finite constant $\Gamma_\Omega(n\mathbb{1}) > 0$ is determined in Theorem 3.19.

Moreover, these generalised superfunctions extend as continuous functionals to the space of all superfunctions $f \in \Gamma(\mathcal{O}_{W,W_{\bar{0},\mathbb{R}}})$ which satisfy Schwartz estimates along $\mathrm{Herm}^+(p) \times 0$*,* i.e.

$$\sup_{z\in\mathrm{Herm}^+(p)} \bigl|(1 + \|z\|)^N f(D; z, w)\bigr| < \infty$$

for all $w \in \mathrm{Herm}(q)$, $N \in \mathbb{N}$, and $D \in S(W)$.

This immediately implies the following explicit formula, which is a precise statement of the identity proved in Ref. [20].

Corollary 2.7. *Let f be a (holomorphic) superfunction on the open subspace of $L(W)$ whose underlying set is the tube $T(\gamma_0)$ from Equation* (2.1)*, satisfying the estimate in Equation* (2.2) *for some $R > 0$ and any $D \in S(W)$, $w \in \mathbb{C}^{q\times q}$, and $N \in \mathbb{N}$. Then*

$$\int_{L(V,V_{\bar{0},\mathbb{R}})} |Dv|\, Q^\sharp(f)(v) = \frac{\sqrt{\pi}^{np}}{\Gamma_\Omega(n\mathbb{1})} \int_\Omega |Dy|\, \mathrm{Ber}(y)^n f(y).$$

Proof of Theorem 2.6. Throughout the proof, we will consider the *cs* manifolds $L(V, V_{\bar{0},\mathbb{R}})$ and $L(W, W_{\bar{0},\mathbb{R}})$ as embedded in $L(U, U_{\bar{0},\mathbb{R}})$.

By Proposition 2.5, the Laplace transform of $Q_\sharp(|Dv|)$ is defined and holomorphic on $T(\gamma)$, where γ is the open subspace of $L(W, W_{\bar{0},\mathbb{R}})$ whose underlying set is

$$\gamma_0 \coloneqq \mathrm{Herm}^+(p) \times \mathrm{Herm}(q).$$

For $x \in_S T(\gamma)_{cs}$, we compute

$$\mathscr{L}(Q_\sharp(|Dv|)(x) = \int_{L(V,V_{\bar{0},\mathbb{R}})} |Dv|\, Q^\sharp(e^{-\mathrm{str}(x\cdot)}) = \int_{L(V,V_{\bar{0},\mathbb{R}})} |Dv|\, e^{-\mathrm{str}(xQ(v))}.$$

For each $x \in W$, we define a linear map $\phi_x : V \to V$ by

$$\phi_x \begin{pmatrix} 0 & a' \\ a & 0 \end{pmatrix} \coloneqq \begin{pmatrix} 0 & a'x \\ xa & 0 \end{pmatrix},$$

for $\left(\begin{smallmatrix} 0 & a' \\ a & 0 \end{smallmatrix}\right) \in V$. By the cyclicity of the supertrace, notice that

$$2\,\mathrm{str}(xQ(v)) = 2\,\mathrm{str}(xaa') = \mathrm{str}(a'xa) + \mathrm{str}(aa'x) = \mathrm{str}(v\phi_x(v)).$$

Now, let $\gamma^+ \subseteq \gamma$ be the open subspace corresponding to

$$\gamma_0^+ \coloneqq \mathrm{Herm}^+(p) \times \mathrm{Herm}^+(q)$$

and let $x \in_S \gamma^+$. Then we may choose $x^{1/2} \in_S \gamma^+$ such that $(x^{1/2})^2 = x$ and make a change of coordinates $v \mapsto \phi_{x^{-1/2}}(v)$. The Berezinian of this coordinate transformation is given by $\mathrm{Ber}(x)^{-n}$ and $\mathrm{str}(v\phi_x(v)) \mapsto \mathrm{str}(v^2)$. So,

$$\mathscr{L}(Q_\sharp(|Dv|))(x) = \mathrm{Ber}(x)^{-n} \int_{L(V,V_{\bar{0},\mathbb{R}})} |Dv|\, e^{-\frac{1}{2}\mathrm{str}(v^2)}.$$

By holomorphicity, both sides of this equation coincide on $T(\gamma) \cap \mathrm{GL}(p|q, \mathbb{C})$.

To determine the Gaussian integral, we pick coordinates (x, ξ, η) on V as follows:

$$\begin{array}{c c} & \begin{array}{ccc} n & p & q \end{array} \\ \begin{array}{c} n \\ p \\ q \end{array} & \begin{pmatrix} 0 & \overline{x_{ji}} & \eta_{ji} \\ x_{ij} & 0 & 0 \\ \xi_{ij} & 0 & 0 \end{pmatrix}, \end{array}$$

in which case

$$-\frac{1}{2}\operatorname{str}(v^2) = -\operatorname{tr}(xx^*) + \operatorname{tr}(\xi\eta) = -\sum_{i=1}^{p}\sum_{j=1}^{n}|x_{ij}|^2 + \sum_{k=1}^{q}\sum_{j=1}^{n}\xi_{kj}\eta_{jk}.$$

The Berezin integral is performed by picking the degree nq term in the expansion of the exponential function for $e^{\operatorname{tr}(\xi\eta)}$, which is just 1 (for a suitable choice of signs).

Further, the remaining integral is just np copies of the Gaussian integral, which contributes $\sqrt{\pi}^{np}$. Therefore, we find

$$\mathscr{L}(Q_\sharp(|Dv|))(x) = \sqrt{\pi}^{np}\operatorname{Ber}(x)^{-n},$$

for any $x \in_S T(\gamma) \cap \mathrm{GL}(p|q, \mathbb{C})$.

On the other hand, the Laplace transform of T_n is also defined and holomorphic on $T(\gamma)$, as follows from the the definition of $\gamma^\circ_{\mathscr{S}}(T_n)$ and the remarks made after the definition of T_n, in conjunction with Theorem C.34. For $x \in_S T(\gamma)$ such that all principal minors of x are invertible, we find

$$\mathscr{L}(T_n)(x) = \int_\Omega |Dy|\operatorname{Ber}(y)^n e^{-\operatorname{str}(xy)} = \Gamma_\Omega(n\mathbb{1})\operatorname{Ber}(x)^{-n},$$

as a special case of Corollary 3.18 below. By Theorem 3.19, the meromorphic function Γ_Ω has neither a pole nor a zero for $n \geqslant p$.

Therefore, the result follows from the injectivity of the Laplace transform in Theorem C.34, in combination with Corollary C.35. □

3. Laplace transforms of conical superfunctions

As we have seen in Section 2, the superbosonisation identity reduces to computing the Laplace transform of both sides. Whereas for the left hand side, this amounts to the evaluation of a standard Gaussian integral, the computation on the right hand side is more intricate.

Following the procedure from the classical case, where $q = 0$, we compute the Laplace transform of certain, more general conical superfunctions. The outcome for $q > 0$ is more complicated than in the classical case, where the Laplace transform has poles, but no zeros; this is quite different for $q > 0$.

3.1. **The conical superfunctions.** We introduce the basic objects of this section, the conical superfunctions. These are a natural generalisation, to the superspace Ω, of the conical polynomials encountered in the theory of Riemannian symmetric spaces. Our main reference to the subject will be the book of Faraut–Korányi [9], which contains a beautiful, elementary, and self-contained account of the theory for symmetric cones.

Let N^+ be the closed subsupergroup of $K_\mathbb{C}$ whose functor of points is defined by

$$N^+(S) := \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in_S K_\mathbb{C} \,\middle|\, A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

That this functor is represented by a closed, connected complex analytic subsupergroup of $K_\mathbb{C}$ is immediate from the implicit function theorem.

Similarly, we define $T_{\mathbb{C}}$ to be the complex supergroup representing the functor

$$T_{\mathbb{C}}(S) := \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in_S K_{\mathbb{C}} \;\middle|\; A = \operatorname{diag}(a), D = \operatorname{diag}(d), a, d \in_S (\mathbb{C}^\times)^{p+q} \right\}.$$

Then $T_{\mathbb{C}}$ normalises N^+, and the subsupergroup $B := T_{\mathbb{C}} N^+$ of $K_{\mathbb{C}}$ generated by $T_{\mathbb{C}}$ and N^+ is again closed and connected. Its complex super-dimension is

$$\dim_{\mathbb{C}} B = (p+q)|0 + \dim_{\mathbb{C}} W.$$

As an immediate consequence of these definitions, we obtain the following lemma.

Lemma 3.1. *The orbit $K_{\mathbb{C}}.1 = \mathrm{GL}(p|q, \mathbb{C})$ is open in W, and the action of B on $K_{\mathbb{C}}$ also admits an open orbit, namely, $B.1$. Here, 1 denotes the identity matrix, considered as an ordinary point of W.*

Proof. The isotropy of the $K_{\mathbb{C}}$-action at 1 is the supergroup representing the functor

$$K_{\mathbb{C},1}(S) := \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\middle|\; A = D \right\} \cong \mathrm{GL}(p|q, \mathbb{C})(S).$$

Since $\dim_{\mathbb{C}} K_{\mathbb{C}} = 2 \dim_{\mathbb{C}} W$ and $\dim_{\mathbb{C}} K_{\mathbb{C},1} = \dim_{\mathbb{C}} W$, the orbit has full dimension, and is therefore open in W.

On the other hand, the isotropy of B represents the functor

$$\big(B \times_{K_{\mathbb{C}}} K_{\mathbb{C},1}\big)(S) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in_S T_{\mathbb{C}} \;\middle|\; A = D \right\},$$

and thus has dimension $(p+q)|0$. Hence, $B.1$ is also open in W.

Finally, it is obvious that

$$(K_{\mathbb{C}}.1)_0 = (K_{\mathbb{C}})_0.1 = \mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}),$$

so in particular, $K_{\mathbb{C}}.1 = \mathrm{GL}(p|q, \mathbb{C})$. $\square$

Let B_p and B_q denote the Borel subgroups of $\mathrm{GL}(p, \mathbb{C})$ and $\mathrm{GL}(q, \mathbb{C})$, respectively, given by lower triangular matrices. Denote their opposite Borels with bars. Then the underlying Lie group of B is $B_0 = B_p \times B_q \times \overline{B}_p \times \overline{B}_q$. Hence, $B.1$ is the open subspace of W corresponding to the open subset $B_p \overline{B}_p \times B_q \overline{B}_q$. This justifies calling $B.1$ the *big cell* of $K_{\mathbb{C}}.1 = \mathrm{GL}(p|q, \mathbb{C})$; it also shows that $(B.1)_0$ is Zariski open, so that it makes sense to speak of regular superfunctions on $B.1$.

We will now define a family of rational superfunctions $\Delta_1, \dots, \Delta_{p+q}$ which in some sense are fundamental (relative) invariants. For any $Z = (Z_{ij}) \in_S W = \mathfrak{gl}(p|q, \mathbb{C})$ and $1 \leqslant k, \ell \leqslant p+q$, we consider to that end

$$[Z]_{k\ell} := (Z_{ij})_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell} \quad \text{and} \quad [Z]_k := [Z]_{kk},$$

so that $[Z]_k$ is the kth principal minor of Z.

Whenever $[Z]_k$ is invertible, we define

$$\Delta_k(Z) := \operatorname{Ber}([Z]_k).$$

This uniquely determines a rational superfunction $\Delta_k \in \mathbb{C}(W)$. We also consider the (even) characters $\chi_k : T_{\mathbb{C}} \to \mathbb{C}^\times$, defined by

$$\chi_k(t) := \frac{\prod_{j=1}^{\min(k,p)} a_j^{-1} d_j}{\prod_{j=p+1}^{\min(k,p+q)} a_j^{-1} d_j} \quad \text{for all} \;\; t = \begin{pmatrix} \operatorname{diag}(a) & 0 \\ 0 & \operatorname{diag}(d) \end{pmatrix} \in_S T_{\mathbb{C}}.$$

Note that $\chi_k(t) = \Delta_k(t^{-1}.1)$. In view of the isomorphism $T_{\mathbb{C}} \cong B/N^+$, we may consider χ_k as character of B.

Proposition 3.2. *Let $1 \leqslant k \leqslant p+q$. Then the rational superfunction Δ_k is regular and invertible on the big cell $B.1$, and*

$$\Delta_k(b^{-1}.Z) = \chi_k(b)\Delta_k(Z), \tag{3.1}$$

for any $b \in_S B$ and $Z \in_S B.1$.

In the *proof* of the proposition, we will need the following lemma.

Lemma 3.3. *Let $Z \in_S W$. Then $Z \in_S B.1$ if and only if $[Z]_k$ is invertible for every $1 \leqslant k \leqslant p+q$.*

Proof. We write

$$Z = \begin{array}{c c} & \begin{smallmatrix} p+q-1 & 1 \end{smallmatrix} \\ \begin{smallmatrix} p+q-1 \\ 1 \end{smallmatrix} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

where $A = [Z]_{p+q-1}$. If A is invertible, we may perform a block decomposition of Z as as follows:

$$Z = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}.$$

In this decomposition, $D - CA^{-1}B$ is automatically an invertible element of $\Gamma(\mathcal{O}_{S,\bar{0}})$ whenever $Z = [Z]_{p+q}$ is invertible.

Provided that all principal minors are invertible, we may, by replacing Z by A, continue with this procedure to arrive at a decomposition of the form $Z = ldu$ where l is strictly lower triangular, d is diagonal, and u is strictly upper triangular. Then $b := (ld, u^{-1}) \in_S B$, and $b.1 = Z$. This shows that the set of all $Z \in_S W$ such that all principal minor are invertible is contained in $(B.1)(S)$.

Given any $Z \in_S W = \mathfrak{gl}(p|q, \mathbb{C})$ such that $[Z]_k$ is invertible, write

$$Z = \begin{array}{c c} & \begin{smallmatrix} k & p+q-k \end{smallmatrix} \\ \begin{smallmatrix} k \\ p+q-k \end{smallmatrix} & \begin{pmatrix} A & * \\ * & * \end{pmatrix} \end{array},$$

where $A = [Z]_k$. Define integers

$$m := \min(p, k) \quad \text{and} \quad n := \min(0, p+q-k),$$

and let $\alpha, \delta \in_S \mathrm{GL}(m|n, \mathbb{C})$. Then

$$\begin{pmatrix} \alpha & 0 \\ * & * \end{pmatrix} \begin{pmatrix} A & * \\ * & * \end{pmatrix} \begin{pmatrix} \delta & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} \alpha A \delta & * \\ * & * \end{pmatrix}, \tag{3.2}$$

where the upper left block of the latter matrix is invertible.

In particular, the open subspace of W whose S-valued points are the $Z \in_S W$ with all principal minors invertible is invariant under the action of B. But this already shows the equality. ☐

Proof of Proposition 3.2. Lemma 3.3 shows that the functions Δ_k are regular on $B.1$. We have already noted the identity

$$\Delta_k(t^{-1}.1) = \chi_k(t)$$

for $t \in_S T_{\mathbb{C}}$. Since, as we have shown in the course of the proof of Lemma 3.3, every $Z \in_S B.1$ may be written as $Z = b.1$ for some $b \in_S B$, it will be sufficient to show that Δ_k is N^+-invariant.

So, let $n = \left(\begin{smallmatrix} n' & 0 \\ 0 & n'' \end{smallmatrix}\right) \in_S N^+$ and $1 \leqslant k \leqslant p+q$. Denote the upper left $k \times k$ block of n' resp. n'' by α and δ, respectively. Then α and δ are invertible. Moreover, Equation (3.2) shows that

$$[n^{-1}.Z]_k = \alpha^{-1}[Z]_k\delta,$$

and hence, that

$$\Delta_k(n^{-1}.Z) = \mathrm{Ber}(\alpha)^{-1}\Delta_k(Z)\mathrm{Ber}(\delta) = \Delta_k(Z),$$

due to the tridiagonal nature of α and δ. This proves the claim. $\square$

Remark 3.4. In case $q = 0$, the functions $\Delta_1, \dots, \Delta_p$ are known from the theory of Jordan algebras. The statement corresponding to Proposition 3.2 is to be found, *e.g.* in [9, Proposition VI.3.10]. In this case, the Δ_k are all polynomials.

In general, this continues to hold Δ_k, $1 \leqslant k \leqslant p$. As for the other Δ_k, $k > p$, they are certainly regular on the larger domain where only $[Z]_{p+1}, \dots, [Z]_{p+q}$ are invertible. By density of the big cell in W, Equation (3.1) continues to hold there.

The rational characters of B (equivalently, of $T_{\mathbb{C}}$) are exactly the superfunctions

$$\chi_{\mathbf{m}} := \chi_1^{m_1-m_2} \cdots \chi_{p+q-1}^{m_{p+q-1}-m_{p+q}} \cdot \chi_{p+q}^{m_{p+q}},$$

where $\mathbf{m} = (m_1, \dots, m_{p+q}) \in \mathbb{Z}^{p+q}$. Explicitly, for $t = (\mathrm{diag}(a), \mathrm{diag}(d)) \in_S T_{\mathbb{C}}$:

$$\chi_{\mathbf{m}}(t) = \frac{\prod_{j=1}^{p}(a_j^{-1}d_j)^{m_j}}{\prod_{j=1}^{q}(a_{p+j}^{-1}d_{p+j})^{m_{p+j}}}.$$

If we define $\Delta_{\mathbf{m}}$ by

$$\Delta_{\mathbf{m}} := \Delta_1^{m_1-m_2} \cdots \Delta_{p+q-1}^{m_{p+q-1}-m_{p+q}} \Delta_{p+q}^{m_{p+q}},$$

then $\Delta_{\mathbf{m}}$ is a regular superfunction on the big cell by Proposition 3.2, and by the same token, we have

$$\Delta_{\mathbf{m}}(b^{-1}.Z) = \chi_{\mathbf{m}}(b)\Delta_{\mathbf{m}}(Z)$$

for all $b \in_S B$ and $Z \in_S B.1$.

Definition 3.5 (Conical superfunctions)**.** A rational function $f \in \mathbb{C}(W)$ is *conical* if its domain of definition is B-invariant and there exists $\mathbf{m} \in \mathbb{Z}^{p+q}$ such that

$$f(b^{-1}.Z) = \chi_{\mathbf{m}}(b)f(Z)$$

for all $b \in_S B$ and all S-valued points Z of the domain of definition of f.

Lemma 3.6. *Let f be a conical function. Then f is proportional to $\Delta_{\mathbf{m}}$ for some multi-index $\mathbf{m} \in \mathbb{Z}^{p+q}$.*

Proof. The domain of definition of f is Zariski open and dense, as is the big cell $B.1$. Hence, f is regular on $B.1$ and uniquely determined by its values $f(Z)$ for any $Z \in_S B.1$ (and any S). Then

$$f(b.1) = \chi_{\mathbf{m}}(b^{-1})f(1) = f(1)\Delta_{\mathbf{m}}(b.1)$$

for any $b \in_S B$, which already shows that $f = f(1)\Delta_{\mathbf{m}}$. $\square$

In one instance below, it will be useful to have an alternative parametrisation of the 'boson-boson sector' of B. To that end, define

$$\tau_j(u) := \begin{array}{c c} & \begin{array}{ccc} \scriptstyle j-1 & \scriptstyle 1 & \scriptstyle p-j \end{array} \\ \begin{array}{c} \scriptstyle j-1 \\ \scriptstyle 1 \\ \scriptstyle p-j \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

for any $1 \leqslant j \leqslant p$ and $u \in \mathbb{C}^{1\times(p-j)}$. Then

$$e^{\tau_k(u)} = \begin{array}{c c} & \begin{array}{ccc} \scriptstyle j-1 & \scriptstyle 1 & \scriptstyle p-j \end{array} \\ \begin{array}{c} \scriptstyle j-1 \\ \scriptstyle 1 \\ \scriptstyle p-j \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \end{array} \in \overline{B}_p.$$

Let u be a Hermitian matrix. We denote the diagonal entries by $u_j = u_{jj}$ and the rows of the upper triangle by

$$u^{(j)} := (u_{j,j+1}, \ldots, u_{jp}) \quad \text{for all} \ \ 1 \leqslant j < p.$$

If $u_1, \ldots, u_p > 0$, the we define

$$t(u) := \delta_1(u_1) e^{\tau_1(u^{(1)})^*} \delta_2(u_2) e^{\tau_2(u^{(2)})^*} \cdots e^{\tau_{p-1}(u^{(p-1)})^*} \delta_p(u_p) \in \mathrm{GL}(p, \mathbb{C}),$$

where we set

$$\delta_j(\lambda) := (\underbrace{1, \ldots, 1}_{j-1}, \lambda, \underbrace{1, \ldots, 1}_{p-j}).$$

Considering as usual $\mathrm{GL}(p, \mathbb{C})$ as a subgroup of H, *via* the map

$$a \mapsto \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (a^*)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we see immediately that for any $Z \in_S B.1$ and $\mathbf{m} \in \mathbb{Z}^{p+q}$, we have

$$\Delta_{\mathbf{m}}(t(u).Z) = \Delta_{\mathbf{m}}(Z).$$

On the other hand, we obtain coordinates on $\mathrm{Herm}^+(p)$ in this fashion, as explained in the proposition below, which translates [9, Proposition VI.3.8] to this special case. We give the direct proof using matrices both for the reader's convenience, and because we will need to refer to it later on.

Proposition 3.7. *Let $z \in \mathrm{Herm}^+(p)$. There is a unique $u \in \mathrm{Herm}^+(p)$, with diagonal entries $u_1, \ldots, u_p > 0$, such that $z = t(u).1$, and it is determined by*

$$z_{jj} = u_j^2 + \sum_{k=1}^{j-1} |u_{kj}|^2 \quad \textit{and} \quad z_{jk} = u_j u_{jk} + \sum_{\ell=1}^{j-1} u_{j\ell} u_{\ell k}$$

for all $1 \leqslant j \leqslant p$ and $j < k$, $k \leqslant p$.

Proof. The proof is by induction on p. For $p = 1$, the statement is trivial. For general p,

$$z = \delta_1(u_1) e^{\tau_1(u^{(1)})^*} \begin{pmatrix} 1 & 0 \\ 0 & z' \end{pmatrix} = \begin{pmatrix} u_1^2 & u_1 u^{(1)} \\ * & z' + (u^{(1)})^* u^{(1)} \end{pmatrix},$$

where u_1 and $u^{(1)}$ are determined by

$$z_{11} = u_1^2 \quad \text{and} \quad z_{1k} = u_1 u_{1k}$$

for all $k > 1$. By the inductive hypothesis, we have $z' = t(u').1 = (z_{jk})_{2 \leqslant j,k \leqslant p}$ for $u' = (u_{jk})_{2 \leqslant j,k \leqslant p}$, where z' and u' are related by

$$z'_{jj} = u_j^2 + \sum_{k=2}^{j-1} |u_{kj}|^2 \quad \text{and} \quad z'_{jk} = u_j u_{jk} + \sum_{\ell=2}^{j-1} u_{j\ell} u_{\ell k}$$

for all $2 \leqslant j \leqslant p$ and $j < k$, $k \leqslant p$. The assertion follows by noting simply that

$$\big((u^{(1)})^* u^{(1)}\big)_{jk} = \overline{u_{1j}} u_{1k} = u_{j1} u_{1k},$$

where we take the entries of this matrix to be indexed over the set $\{2, \ldots, p\}$. $\square$

3.2. **The invariant Berezinian.** Recall the definition of the homogeneous superspace $\Omega = H/H \cap K \cong H.1$ from Section 2. The underlying manifold is

$$\Omega_0 = \mathrm{Herm}^+(p) \times \mathrm{U}(q),$$

where

$$\mathrm{Herm}^+(p) = \mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p)$$

is the cone of positive definite Hermitian $p \times p$ matrices. We observe that the dimension of the *cs* manifold Ω coincides with the graded dimension of the complex supermanifold $L(W)$. In particular, since Ω_0 is contained in $(B.1)_0$, it follows that Ω is a subspace of the *cs* manifold associated with the complex supermanifold $B.1$.

On general grounds [1, 2], Ω admits a non-zero H-invariant Berezinian density, unique up to a constant. Due to the special features of this example, we can give an explicit formula. To that end, recall the facts and definitions from Appendix B.

Observe that Ω has purely even codimension in $L(W)_{cs}$. Thus, the standard coordinates

$$Z = \begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} z & \zeta \\ \omega & w \end{pmatrix} \end{array}$$

on $L(W)_{cs}$ give a system of coordinates on Ω. This defines a retraction of Ω, which we call *standard*, and a system of adapted fibre coordinates for this retraction.

In particular, $D(\zeta, \omega)$ is a well-defined relative Berezinian (density) on Ω over Ω_0, with respect to the standard retraction. Denote by $|dz|$ the Lebesgue density on $\mathrm{Herm}(p)$, and by $|dw|$ the normalised invariant density on $\mathrm{U}(q)$. We set

$$d\mu(Z) := \frac{|dz|\,|dw|}{|\det z|^p}\, D(\zeta, \omega)\, \det(z - \zeta w^{-1}\omega)^q \det(w - \omega z^{-1}\zeta)^p.$$

Proposition 3.8. *The Berezinian density μ is H-invariant.*

We divide the *proof* of this statement into several lemmata which will also be useful below when computing Laplace transforms. Recall the notion of nilpotent shifts from Appendix A.3.

Lemma 3.9. *Let n be a nilpotent shift for $\mathrm{U}(q) = \mathrm{U}(q) \times \mathrm{U}(q)/\operatorname{diag} \subseteq \mathbb{C}^{q\times q}$. Then*

$$\int_{\mathrm{U}(q)} |dw|\, f(w + n) = \int_{\mathrm{U}(q)} |dw|\, f(w) \frac{\det(w)^q}{\det(w - n)^q}$$

for any smooth function f on $\mathrm{U}(q)$.

Proof. The invariant density on $\mathrm{U}(q)$ coincides, due to the invariance, with the Riemannian density for any invariant Riemannian metric. By Lemma B.1, we have

$$\int_{\mathrm{U}(q)} |dw|\, f(w + n) = \int_{\mathrm{U}(q)} |dw|\, f J_1$$

where J_1 is determined by $J_0 = 1$ and

$$\frac{d}{dt} \log J_t(w) = -\operatorname{div} v_n(w - tn) = \tfrac{1}{2} \operatorname{tr}_{\mathfrak{gl}(q)} R_{(w-tn)^{-1}w} = q \operatorname{tr}((w - tn)^{-1}w),$$

for $w \in \mathrm{U}(q)$, since for $u \in \mathfrak{u}(q)$ and $w \in \mathbb{C}^{q\times q}$, we have

$$R_w(u) = \frac{d}{dt} \exp(tu) w \exp(tu)\Big|_{t=0} = uw + wu.$$

Setting

$$J_t(w) := \frac{\det(w)^q}{\det(w - tn)^q}$$

manifestly solves the equation. $\square$

Lemma 3.10. *Let n be a nilpotent shift for* $\mathrm{Herm}^+(p) = \mathrm{GL}(p,\mathbb{C})/U(p) \subseteq \mathbb{C}^{q\times q}$. *Then*

$$\int_{\mathrm{Herm}^+(p)} \frac{|dz|}{|\det z|^p}\, f(z+n) = \int_{\mathrm{U}(q)} \frac{|dz|}{|\det z|^p}\, f(z)\frac{\det(z)^p}{\det(z-n)^p}$$

for any compactly supported smooth function f on $\mathrm{Herm}^+(p)$.

Proof. Again Lemma B.1 applies, since the invariant density is the Riemannian density, and we have

$$\int_{\mathrm{Herm}^+(p)} \frac{|dz|}{|\det z|^p}\, f(z+n) = \int_{\mathrm{Herm}^+(p)} \frac{|dz|}{|\det z|^p}\, f J_1$$

where J_1 is determined by $J_0 = 1$ and

$$\begin{aligned}\frac{d}{dt}\log J_t(z) &= -\operatorname{div} v_n(z-tn) = \tfrac{1}{2}\operatorname{tr}_{\mathfrak{gl}(p)} R_{(z-tn)^{-1/2}z(z-tn)^{-1/2}}\\ &= p\operatorname{tr}((z-tn)^{-1/2}z(z-tn)^{-1/2}) = p\operatorname{tr}((z-tn)^{-1}z),\end{aligned}$$

for $z \in \mathrm{Herm}^+(p)$, since for $u \in \mathrm{Herm}(p)$ and $z \in \mathbb{C}^{p\times p}$, we have

$$R_w(u) = \frac{d}{dt}\exp(tu)w\exp(tu)\Big|_{t=0} = uw + wu.$$

Hence, all we have to do is to solve the same initial value problem as in the proof of Lemma 3.9, with q replaced by p. ☐

Now, for $Z \in_S K_{\mathbb{C}}.1$, set

$$\varrho(Z) := \det(z - \zeta w^{-1}\omega)^q \det(w - \omega z^{-1}\zeta)^p.$$

Observe that

$$\varrho(Z) = \mathrm{Ber}(Z)^{q-p}\det(z)^p\det(w)^q = \Delta_{q\mathbb{1}'+(q-p)\mathbb{1}''}(Z)\det(w)^q, \tag{3.3}$$

where we define

$$\mathbb{1}' := (\underbrace{1,\dots,1}_{p},\underbrace{0,\dots,0}_{q}) \quad\text{and}\quad \mathbb{1}'' := (\underbrace{0,\dots,0}_{p},\underbrace{1,\dots,1}_{q}).$$

For $h \in_S K_{\mathbb{C}}$, we define

$$I_h^f(z,w) := \int_{\mathbb{C}^{0|p\times q}\oplus\mathbb{C}^{0|q\times p}} D(\zeta,\omega)\,\varrho(Z)f(h.Z)$$

whenever $f(h.Z)$ is defined.

Lemma 3.11. *Let $f \in \Gamma(\mathcal{O}_\Omega)$ and $h = \left(\begin{smallmatrix} A & 0\\ 0 & D^{-1}\end{smallmatrix}\right)$, where $A = \left(\begin{smallmatrix} 1 & 0\\ \alpha & 1\end{smallmatrix}\right) \in_S \mathrm{GL}(p|q,\mathbb{C})$ and $D = \left(\begin{smallmatrix} 1 & \delta\\ 0 & 1\end{smallmatrix}\right) \in_S \mathrm{GL}(p|q,\mathbb{C})$. Then for any smooth function φ on $\mathrm{U}(q)$, we have*

$$\int_{\mathrm{U}(q)} |dw|\,\varphi(w)I_h^f(z,w) = \int_{\mathrm{U}(q)} |dw| \int D(\zeta,\omega)\,\varrho(Z)f(Z)\varphi(w-\alpha\zeta-\omega\delta-\alpha z\delta).$$

Proof. Firstly, note that the conditions set out above imply that $h \in_S H$, so that the statement of the lemma is meaningful. We have

$$h.Z = AZD = \begin{pmatrix} z & \zeta + z\delta\\ \omega+\alpha z & w+\alpha\zeta+\omega\delta+\alpha z\delta\end{pmatrix}.$$

By the use of the coordinate change $\zeta \mapsto \zeta + z\delta$, $\omega \mapsto \omega + \alpha z$, we find

$$I_h^f(z,w) = \int D(\zeta,\omega)\,\varrho\begin{pmatrix} z & \zeta - z\delta\\ \omega-\alpha z & w\end{pmatrix} f\begin{pmatrix} z & \zeta\\ \omega & w+\alpha\zeta+\omega\delta+\alpha z\delta\end{pmatrix}.$$

Applying Lemma 3.9 with the nilpotent shift $n = \alpha\zeta + \omega\delta + \alpha z\delta$, we see that the left hand side $\int |dw|\,\varphi(w)I_h^f(z,w)$ equals

$$\int D(\zeta,\omega)\int |dw|\,\varrho(h^{-1}.Z)f(Z)\frac{\varphi(w-\alpha\zeta-\omega\delta-\alpha z\delta)\det(w)^q}{\det(w-\alpha\zeta-\omega\delta-\alpha z\delta)^q}.$$

By appealing to Equation (3.3), the claim follows. □

Lemma 3.12. *Let $f \in \Gamma_c(\mathcal{O}_\Omega)$ and $h = \left(\begin{smallmatrix} A & 0 \\ 0 & D^{-1} \end{smallmatrix}\right)$, where $A = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \in_S \mathrm{GL}(p|q,\mathbb{C})$ and $D = \left(\begin{smallmatrix} 1 & 0 \\ \delta & 1 \end{smallmatrix}\right) \in_S \mathrm{GL}(p|q,\mathbb{C})$. For any smooth function φ on $\mathrm{Herm}^+(p)$, we have*

$$\int_{\mathrm{Herm}^+(p)} \frac{|dz|}{|\det z|^p}\,\varphi(z) I_h^f(z,w) = \int \frac{|dz|}{|\det z|^p} \int D(\zeta,\omega)\,\varrho(Z) f(Z) \varphi(z-\alpha\omega-\zeta\delta-\alpha w\delta).$$

Proof. We may proceed similarly as in the proof of Lemma 3.11. Indeed,

$$h.Z = AZD = \begin{pmatrix} z+\alpha\omega+\zeta\delta+\alpha w\delta & \zeta+\alpha w \\ \omega+w\delta & w \end{pmatrix}.$$

The coordinate change $\zeta \mapsto \zeta+\alpha w$, $\omega \mapsto \omega + w\delta$ leads to

$$I_h^f(z,w) = \int D(\zeta,\omega)\,\varrho \begin{pmatrix} z & \zeta-\alpha w \\ \omega - w\delta & w \end{pmatrix} f \begin{pmatrix} z+\alpha\omega+\zeta\delta+\alpha w\delta & \zeta \\ \omega & w \end{pmatrix}.$$

Applying Lemma 3.10 with the nilpotent shift $n = \alpha\omega+\zeta\delta+\alpha w\delta$, we find that the left hand side $\int_{\mathrm{Herm}^+(p)} \frac{|dz|}{|\det(z)|^p}\varphi(z) I_h^f(z)$ equals

$$\int D(\zeta,\omega) \int \frac{|dz|}{|\det(z)^p|}\,\varrho(h^{-1}.Z) f(Z) \frac{\varphi(z-\alpha\omega-\zeta\delta-\alpha w\delta)\det(z)^p}{\det(z-\alpha\omega-\zeta\delta-\alpha w\delta)^p}.$$

As above, this proves the claim, by the use of Equation (3.3). □

Remark 3.13. Observe that the Borel supergroups used in Lemma 3.11 and Lemma 3.12 are opposite.

Proof of Proposition 3.8. If $h \in_S H$, $h = \left(\begin{smallmatrix} A & 0 \\ 0 & D^{-1} \end{smallmatrix}\right)$, is of the form set out in Lemma 3.11 or Lemma 3.12, then by the token of these, μ is invariant under the action of h. Decomposing A and D in the general case into elements of block diagonal and block triangular form, we may thus assume

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Then

$$h.Z = AZD = \begin{pmatrix} a_1 z d_1 & a_1 \zeta d_2 \\ a_2 \omega d_1 & a_2 w d_2 \end{pmatrix}.$$

The coordinate changes $\zeta \mapsto a_1\zeta d_2$ and $\omega \mapsto a_2 \omega d_1$ show that $I_h^f(z,w)$ equals

$$\det(a_1 d_1)^q \det(a_2 d_2)^p \int D(\zeta,\omega)\,\varrho \begin{pmatrix} z & a_1^{-1}\zeta d_2^{-1} \\ a_2^{-1}\omega d_1^{-1} & w \end{pmatrix} f \begin{pmatrix} a_1 z d_1 & \zeta \\ \omega & a_2 w d_2 \end{pmatrix}.$$

Since

$$\begin{aligned} \varrho(A^{-1}ZD^{-1}) &= \mathrm{Ber}(A^{-1}ZD^{-1})^{q-p} \det(a_1^{-1} z d_1^{-1})^p \det(a_2^{-1} w d_2^{-1})^q \\ &= \det(a_1 d_1)^{p-q} \det((a_1 d_1)^{-1})^p \det(a_2 d_2)^{q-p} \det((a_2 d_2)^{-1})^q \varrho(Z) \\ &= \det(a_1 d_1)^{-q} \det(a_2 d_2)^{-p} \varrho(Z), \end{aligned}$$

it follows, by applying the invariance of the densities on $\mathrm{Herm}^+(p)$ and $\mathrm{U}(q)$, that μ is invariant under the action of h. This proves the proposition. □

3.3. **The Laplace transform of conical superfunctions.** We now come finally to the core of our paper, the explicit computation of the Laplace transforms of conical superfunctions. We will make heavy use of the facts and definitions laid down in Appendix B.

Fix a superfunction $f \in \Gamma(\mathcal{O}_\Omega)$ and $x \in_S L(W)_{cs}$. Whenever the integral converges, we define the *Laplace transform* of f at x by

$$\mathscr{L}(f)(x) \coloneqq \int_\Omega |Dy|\, e^{-\operatorname{str}(xy)} f(y),$$

where we write $|Dy|$ for the invariant Berezinian μ on Ω. All integrals will be taken with respect to the standard retraction on Ω.

Proposition 3.14. *For $x \in_S B.1$, the integral*

$$\mathscr{L}(\Delta_{\mathbf{m}})(x^{-1}) = \int_\Omega |Dy|\, e^{-\operatorname{str}(x^{-1}y)} \Delta_{\mathbf{m}}(y)$$

converges absolutely if and only if $m_j > j-1$ for $j = 1, \ldots, p$.

We make use of the following lemma.

Lemma 3.15. *Define*

$$I_x\left(\begin{smallmatrix} z & 0 \\ 0 & w \end{smallmatrix}\right) \coloneqq \int_{\mathbb{C}^{0|p\times q}\oplus\mathbb{C}^{0|q\times p}} D(\zeta,\omega)\, \varrho(y) e^{-\operatorname{str}(x^{-1}y)} \Delta_{\mathbf{m}}(y)$$

for $x \in_S \mathrm{GL}(p|q,\mathbb{C})$. Let $h \in_S B_0$. Then

$$I_{h.x}\left(\begin{smallmatrix} z & 0 \\ 0 & w \end{smallmatrix}\right) = \chi_{\mathbf{m}}(h^{-1}) I_x\left(h^{-1}.\left(\begin{smallmatrix} z & 0 \\ 0 & w \end{smallmatrix}\right)\right).$$

Proof. If $h = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \in_S K_{\mathbb{C}}$, then

$$\operatorname{str}((h.x)^{-1}y) = \operatorname{str}((AxD^{-1})^{-1}y) = \operatorname{str}(x^{-1}A^{-1}yD) = \operatorname{str}(x^{-1}(h^{-1}.y)). \tag{3.4}$$

Assume now that $h \in_S B_0$, where $A = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$. Arguing as in the proof of Proposition 3.8 and using Equation (3.3), we obtain

$$I_{h.x}\left(\begin{smallmatrix} z & 0 \\ 0 & w \end{smallmatrix}\right) = \frac{\det(a_1)^q \det(a_2)^p}{\det(d_1)^q \det(d_2)^p} \int D(\zeta,\omega)\, e^{-\operatorname{str}(x^{-1}y)} \Delta_{\mathbf{m}+q\mathbb{1}'+(q-p)\mathbb{1}''}(h.y') \det(w)^q$$

with

$$y' \coloneqq h^{-1}.\begin{pmatrix} z & a_1\zeta d_2^{-1} \\ a_2\omega d_1^{-1} & w \end{pmatrix} = \begin{pmatrix} a_1^{-1} z d_2 & \zeta \\ \omega & a_2^{-1} w d_2 \end{pmatrix}.$$

In view of

$$\chi_{q\mathbb{1}'-p\mathbb{1}''}(h^{-1}) = \det(a_1^{-1}d_1)^q \det(a_2 d_2^{-1})^p,$$

this leads to the desired conclusion immediately. ☐

Proof of Proposition 3.14. Let us first show that the condition stated in the proposition is sufficient for the convergence of the integral.

To that end, we perform the coordinate change $u \mapsto z = z(u) = t(u).1$ by the aid of Proposition 3.7. The pullback of $|dz|$ is $2^p \prod_{j=1}^p u_j^{2(p-j)+1}\, |du|$, where, using the short-hand $|du_{jk}| = |d\Re u_{jk}||d\Im u_{jk}|$, we write $|du|$ for the Lebesgue density $\prod_{j=1}^p |du_j| \prod_{1\leqslant j<k\leqslant p} |du_{jk}|$. Hence, $|\det z|^{-p}\, |dz|$ pulls back to $2^p \prod_{j=1}^p u_j^{-2j+1}\, |du|$.

Now, let x denote the generic point of $B.1 \cap G'.0$. Writing $x^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we see

$$(t(u)^{-1}.x)^{-1} = \begin{pmatrix} t(u)^* a t(u) & t(u)^* b \\ c t(u) & d \end{pmatrix}.$$

From Lemma 3.15, we obtain

$$I_x(z,w) = \chi_{\mathbf{m}}(t(u)^{-1}) I_{t(u)^{-1}.x}(1,w) = \prod_{j=1}^p u_j^{2m_j} e^{-\operatorname{tr}(t(u)at(u)^*)} f(u,w), \tag{3.5}$$

where

$$f(u,w) \coloneqq e^{\operatorname{tr}(dw)} \det(w)^q \int D(\zeta,\omega)\, \Delta_{\mathbf{m}+q\mathbb{1}'+(q-p)\mathbb{1}''} \left(\begin{smallmatrix} 1 & \zeta \\ \omega & w \end{smallmatrix}\right) e^{-\operatorname{tr}(t(u)^* b\zeta)+\operatorname{tr}(ct(u)\omega)}.$$

Observe that $f \in \Gamma(\mathcal{O}_{\mathrm{U}(q)})[u]$. Here, for any locally convex vector space E, we denote by $E[u]$ the set of polynomials in $u_j, u_{jk}, \overline{u_{jk}}$ with values in E.

Moreover,

$$\operatorname{tr}\bigl(t(u)^* a t(u)\bigr) = \operatorname{tr}\bigl(at(u)t(u)^*\bigr) = \operatorname{tr}(az).$$

To see that the integral $\int_\Omega |Dy|\, e^{-\operatorname{str}(x^{-1}y)} \Delta_{\mathbf{m}}(y)$ converges absolutely, it will thus be sufficient to show that

$$J_{\mathbf{m}}(f,a) \coloneqq \int |du| \prod_{j=1}^p u_j^{2(m_j-j+1)-1} f(u) e^{-\operatorname{tr}(az(u))}$$

converges absolutely for any $f \in \mathbb{C}[u]$, uniformly on compact subsets with all derivatives in $a \in \operatorname{Herm}^+(p) + i \operatorname{Herm}^+(p)$. Since taking derivatives with respect to a only introduces polynomials in u into the integrand, it will be sufficient to show uniform convergence on compact subsets with respect to a.

Thus, let $\alpha = (\alpha_j)_{1\leqslant j\leqslant p} \cup (\alpha_{jk}, \bar\alpha_{jk})_{1\leqslant j<k\leqslant p}$, $\alpha_j, \alpha_{jk} \in \mathbb{N}$, and consider

$$f(u) = u^\alpha \coloneqq \prod_j u_j^{\alpha_j} \prod_{j<k} (u_{jk})^{\alpha_{jk}} (\overline{u_{jk}})^{\bar\alpha_{jk}}.$$

Below, we will use the notation $\alpha^{(j)} = (\alpha_{jk}, \bar\alpha_{jk})_{j<k\leqslant p}$ for any $1 \leqslant j < p$.

We prove the convergence of the integral $J^\alpha_{\mathbf{m}}(a) \coloneqq J_{\mathbf{m}}(u^\alpha, a)$ by induction on p. For $p = 1$, we have

$$J^{\alpha_1}_{m_1}(a_1) = \int_0^\infty |du_1|\, u_1^{2m_1+\alpha_1-1} e^{-\Re(a_1)u_1^2} = \frac{\Gamma\bigl(m_1 + \frac{\alpha_1}{2}\bigr)}{2\Re(a_1)^{m_1+\alpha_1/2}},$$

with uniform convergence on compact subsets of $\Re a_1 > 0$, provided that $m_1 > 0$.

For $p \geqslant 2$, by the proof of Proposition 3.7, we may decompose a, u, and $z(u)$:

$$a = \begin{array}{c c} & \begin{array}{cc} 1 & p-1 \end{array} \\ \begin{array}{c} 1 \\ p-1 \end{array} & \begin{pmatrix} a_1 & a^{(1)} \\ (a^{(1)})^* & a' \end{pmatrix} \end{array}, \quad u = \begin{array}{c c} & \begin{array}{cc} 1 & p-1 \end{array} \\ \begin{array}{c} 1 \\ p-1 \end{array} & \begin{pmatrix} u_1 & u^{(1)} \\ (u^{(1)})^* & u' \end{pmatrix} \end{array},$$

and

$$z(u) = \begin{array}{c c} & \begin{array}{cc} 1 & p-1 \end{array} \\ \begin{array}{c} 1 \\ p-1 \end{array} & \begin{pmatrix} u_1^2 & u_1 u^{(1)} \\ u_1 (u^{(1)})^* & z'(u') + (u^{(1)})^* u^{(1)} \end{pmatrix} \end{array}.$$

Then

$$J^\alpha_{\mathbf{m}}(a) = J^{\alpha'}_{\mathbf{m}'}(a') \int |du^{(1)}||du_1||u^{(1)}|^{\alpha^{(1)}} u_1^{2m_1+\alpha_1-1} e^{-a_1u_1^2 - 2u_1 \Re a^{(1)} u^{(1)*} - \Re u^{(1)} a' u^{(1)*}}.$$

We now write $\Re a'$ for the Hermitian part of the matrix a', which is positive definite by assumption. Setting $\tilde a^{(1)} \coloneqq (\Re a')^{-1/2} a^{(1)}$, we find

$$\begin{aligned} &\int |du^{(1)}||u^{(1)}|^{\alpha^{(1)}} e^{-2u_1 \Re a^{(1)} u^{(1)*} - \Re u^{(1)} a' u^{(1)*}} \\ &\qquad = (\det \Re a')^{-1} \int |du^{(1)}| \left|(\Re a')^{-1/2} u^{(1)}\right|^{\alpha^{(1)}} e^{\|u^{(1)} - \tilde a^{(1)}\|^2 - u_1^2 \|\tilde a^{(1)}\|^2} \\ &\qquad = \frac{e^{-u_1^2\|\tilde a^{(1)}\|^2}}{\det \Re a'} \prod_{j=2}^p \int d|u_{1j}| \left|\bigl((\Re a')^{-1/2} u^{(1)}\bigr)_{1j}\right|^{\alpha_{1j}+\bar\alpha_{1j}} e^{-|u_{1j} - (\tilde a^{(1)})_{1j}|^2}. \end{aligned}$$

The latter integral converges, uniformly on compact subsets as a function of a' and $a^{(1)}$, since $\int_{-\infty}^\infty |dt|\, |t|^{2\beta-1} e^{-t^2} = \Gamma(\beta)$ for $\beta > 0$.

Using the formula from the case $p = 1$, we see that $J^{\alpha}_{\mathbf{m}}(a)$ converges uniformly on compact subsets as a function of a for any α, provided that $m_j > j-1$ for all $j = 1, \dots, p$. This completes the proof of sufficiency.

Necessity follows from [9, Theorem VII.1.1] by setting $x = 1$ in the expression for the integral derived at the beginning of the proof. □

Having established convergence, we may study the behaviour of $\mathscr{L}(\Delta_{\mathbf{m}})(x^{-1})$ as a function of x.

Proposition 3.16. *Assume that $m_j > j-1$ for all $j = 1, \dots, p$. Then the function F defined on S-valued points as $F(x) \coloneqq \mathscr{L}(\Delta_{\mathbf{m}})(x^{-1})$ is conical.*

Proof. Let $v \in \mathfrak{b}$ be an element of the Lie superalgebra of B. Denote by the same letter the vector field induced by v on $B.1$. Following the exposition in Appendix A.2, we may consider v as an S-valued point of $B.1$ for $S \coloneqq (B.1)[\varepsilon, \tau]$; namely, after identifiying $\mathfrak{b}$ with the $*[\varepsilon, \tau]$-valued points of H along $1_H : * \to H$, it is induced from $v \in \mathfrak{b}$ by the action of B.

Recall from Equation (3.4) that we have $\operatorname{str}((h^{-1}.x)^{-1}y) = \operatorname{str}(x^{-1}(h.y))$ for any $h \in_S B$. So we compute, with $h = v$ understood as above, that for $x \in_S B.1$

$$\begin{aligned}\mathcal{L}_v(F)(x) = \mathscr{L}(\Delta_{\mathbf{m}})((v^{-1}.x)^{-1}) &= \int_\Omega |Dy|\, e^{-\operatorname{str}(x^{-1}(v.y))}\Delta_{\mathbf{m}}(y)\\ &= -\int_\Omega |Dy|\, \mathcal{L}_v(e^{-\operatorname{str}(x^{-1}\cdot)})(y)\Delta_{\mathbf{m}}(y),\end{aligned}$$

where we have used Equation (B.1).

By Proposition 3.8, $|Dy|$ is H-invariant, so $\mathcal{L}_u(|Dy|) = 0$ for $u \in \mathfrak{h}$. But $\mathfrak{b} \subseteq \mathfrak{h}$, so $\mathcal{L}_v(|Dy|) = 0$, and we compute

$$\mathcal{L}_v\big(|Dy|\, e^{-\operatorname{str}(x^{-1}\cdot)}\Delta_{\mathbf{m}}\big) = |Dy|\, \big(\mathcal{L}_v(e^{-\operatorname{str}(x^{-1}\cdot)})\Delta_{\mathbf{m}} + e^{-\operatorname{str}(x^{-1}\cdot)}\mathcal{L}_v(\Delta_{\mathbf{m}})\big). \tag{3.6}$$

Since

$$\mathcal{L}_v(\Delta_{\mathbf{m}})(y) = \Delta_{\mathbf{m}}(v^{-1}.y) = \chi_{\mathbf{m}}(v)\Delta_{\mathbf{m}}(y) = d\chi_{\mathbf{m}}(v)\Delta_{\mathbf{m}}(y),$$

we see that the right hand side of Equation (3.6) is absolutely integrable, and hence, so is the left hand side. It follows that

$$\mathcal{L}_v(F) = d\chi_{\mathbf{m}}(v)F.$$

The differential equation

$$\dot\gamma = d\chi_{\mathbf{m}}(v)\gamma$$

with initial condition $\gamma(0) = F(x)$, of which $\gamma(t) = F(\exp(-tv)x)$ is a solution, has values in the Fréchet space $\Gamma(\mathcal{O}_S)$. In general, the solutions of linear ODE with values in Fréchet spaces are not unique (*cf.* Ref. [26]). However, $d\chi_{\mathbf{m}}(v)$ is a scalar, so it induces a continuous endomorphism in the topology on $\Gamma(\mathcal{O}_S)$ generated by any continuous norm from a defining set, and we may apply the uniqueness theorem from the Banach case.

Using the facts that $\exp : \mathfrak{b} \to B$ is a local isomorphism and B is connected, we deduce as in the Lie group case that $F(b^{-1}x) = \chi_{\mathbf{m}}(b)F(x)$ for any $b \in_S B$. □

Definition 3.17 (Gamma function)**.** The gamma function of Ω is defined as:

$$\Gamma_\Omega(\mathbf{m}) \coloneqq \mathscr{L}(\Delta_{\mathbf{m}})(1) = \int_\Omega |Dy| e^{-\operatorname{str}(y)}\Delta_{\mathbf{m}}(y),$$

whenever $m_j > j-1$ for all $j = 1, \dots, p$.

With this notation, the following is immediate.

Corollary 3.18. *Assume $m_j > j-1$ for all $j=1,\dots,p$. Then for all $x \in_S B.1$, we have*

$$\mathscr{L}(\Delta_{\mathbf{m}})(x^{-1}) = \Gamma_\Omega(\mathbf{m})\Delta_{\mathbf{m}}(x).$$

Of course, the value of this corollary depends on the extent to which we have control over Γ_Ω. In fact, we can give an entirely explicit expression, as follows.

Theorem 3.19. *Let $m_j > j-1$ for all $j=1,\dots,p$. We have*

$$\begin{aligned}\Gamma_\Omega(\mathbf{m}) = (2\pi)^{p(p-1)/2} &\prod_{j=1}^{p} \Gamma(m_j-(j-1)) \\ &\times \prod_{k=1}^{q} \frac{\Gamma(q-(k-1))}{\Gamma(m_{p+k}+q-(k-1))} \frac{\Gamma(m_{p+k}+k)}{\Gamma(m_{p+k}-p+k)}.\end{aligned}$$

In particular, $\Gamma_\Omega(\mathbf{m})$ extends uniquely as a meromorphic function of $\mathbf{m} \in \mathbb{C}^{p+q}$, and it has neither zeros nor poles if

$$\begin{aligned} m_j &> j-1 \qquad j=1,\dots,p, \\ m_{p+k} &> p-k \qquad k=1,\dots,q. \end{aligned}$$

If, under this assumption, $\mathbf{m}$ is a double partition, then it is a hook partition.

Proof. Our stategy of proof is to reduce the computation to three separate computations, which take place on the fermionic part $\mathbb{C}^{0|p\times q}\oplus\mathbb{C}^{0|q\times p}$, the 'fermion-fermion sector' $\mathrm{U}(q)$, and on the 'boson-boson sector' $\mathrm{Herm}^+(p)$, respectively.

Let us decompose $\mathbf{m} = (\mathbf{m}', \mathbf{m}'')$ where

$$\mathbf{m}' = (m_1,\dots,m_p) \quad \text{and} \quad \mathbf{m}'' = (m_{p+1},\dots,m_{p+q}).$$

Then from Equation (3.5), we have

$$\int D(\zeta,\omega)\, e^{-\operatorname{str}(y)} \Delta_{\mathbf{m}+q\mathbb{1}'+(q-p)\mathbb{1}'}(y) \det(w)^q = e^{-\operatorname{tr}(z)+\operatorname{tr}(w)} \Delta_{\mathbf{m}'}(z)\psi(w),$$

where

$$\psi(w) := \int D(\zeta,\omega)\, \Delta_{\mathbf{m}+q\mathbb{1}'+(q-p)\mathbb{1}''} \begin{pmatrix} 1 & \zeta \\ \omega & w \end{pmatrix} \det(w)^q. \tag{3.7}$$

Appealing to [9, Theorem VII.1.1], we see that

$$\Gamma_\Omega(\mathbf{m}) = (2\pi)^{p(p-1)/2} \prod_{j=1}^{p} \Gamma(m_j-(j-1)) \int_{\mathrm{U}(q)} |dw|\, e^{\operatorname{tr}(w)} \psi(w).$$

The final statement now follows from Lemma 3.20 and Lemma 3.21 below. □

Lemma 3.20. *In the notation from Equation (3.7), we have*

$$\psi(w) = (\Delta_{\mathbf{m}''}(w))^{-1} \prod_{k=1}^{q} \frac{\Gamma(m_{p+k}+k)}{\Gamma(m_{p+k}-p+k)},$$

where $\mathbf{m}'' := (m_{p+1},\dots,m_{p+q})$.

Proof. Notice that $\psi(w)$ is well-defined for w in the open subset $B_q\overline{B}_q \subseteq \mathbb{C}^{q\times q}$, in view of Lemma 3.3. By applying Lemma 3.15 for $x=0$, we see that

$$\psi(h^{-1}.w) = \chi_{\mathbf{m}}(h)\psi(w)$$

for any $h = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \in \mathrm{U}(q) \subseteq B_0$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$. Applying Lemma 3.6 for the case of $p=0$, we find

$$\psi(w) = \Delta_{\mathbf{m}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & w \end{smallmatrix}\right)\psi(1) = (\Delta_{\mathbf{m}''}(w))^{-1}\psi(1).$$

We have

$$\begin{pmatrix}1 & \zeta\\ \omega & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ \omega & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1-\omega\zeta\end{pmatrix}\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix},$$

so

$$\Delta_{\mathbf{m}+q\mathbb{1}'+(q-p)\mathbb{1}''}\begin{pmatrix}1 & \zeta\\ \omega & 1\end{pmatrix} = (\Delta_{\mathbf{m}''+(q-p)\mathbb{1}''}(1-\omega\zeta))^{-1}$$

To calculate the resulting Berezin integral $\psi(1) = \gamma_{-(\mathbf{m}''+(q-p)\mathbb{1}'')}$, where

$$\gamma_{-\mathbf{m}''} \coloneqq \int D(\zeta,\omega)\,(\Delta_{\mathbf{m}''}(1-\omega\zeta))^{-1},$$

we decompose the matrices as

$$\omega = \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle q-1}\end{array}\!\!\begin{array}{c}{\scriptstyle p}\\ \begin{pmatrix}\omega_1\\ \omega'\end{pmatrix}\end{array} \quad\text{and}\quad \zeta = {\scriptstyle p}\begin{array}{cc}{\scriptstyle 1} & {\scriptstyle q-1}\\ \multicolumn{2}{c}{(\ \zeta_1 \quad \zeta'\ )}\end{array}.$$

So, $1-\omega\zeta$ can be decomposed as

$$\begin{pmatrix}1 & 0\\ \omega'\zeta_1(1-\omega_1\zeta_1)^{-1} & 1\end{pmatrix}\begin{pmatrix}1-\omega_1\zeta_1 & 0\\ 0 & 1-\omega'(1+N)\zeta'\end{pmatrix}\begin{pmatrix}1 & (1-\omega_1\zeta_1)^{-1}\omega_1\zeta'\\ 0 & 1\end{pmatrix},$$

where $N = \zeta_1(1-\omega_1\zeta_1)^{-1}\omega_1$. Then $\gamma_{-(\mathbf{m}''+(q-p)\mathbb{1}'')}$ becomes

$$\int D(\zeta_1,\omega_1)\int D(\zeta',\omega')\,\Delta_{\mathbf{m}''+(q-p)\mathbb{1}''}\begin{pmatrix}1-\omega_1\zeta_1 & 0\\ 0 & 1-\omega'(1+N)\zeta'\end{pmatrix}^{-1}$$

Observe that $\operatorname{tr} N = -(1-\omega_1\zeta_1)^{-1}\omega_1\zeta_1$ and hence

$$N^2 = \zeta_1(1-\omega_1\zeta_1)^{-1}\omega_1\zeta_1(1-\omega_1\zeta_1)^{-1}\omega_1 = -\operatorname{tr} N\cdot N.$$

This implies

$$\operatorname{tr}(N^{k+1}) = -\operatorname{tr} N\operatorname{tr}(N^k) = \cdots = -(-\operatorname{tr} N)^{k+1},$$

so that

$$\operatorname{tr}\log(1+N) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\operatorname{tr}\bigl(N^k\bigr) = -\log(1-\operatorname{tr} N).$$

Then

$$\begin{aligned}\det(1+N)^{-1} &= \exp(-\operatorname{tr}\log(1+N)) = 1-\operatorname{tr} N\\ &= 1+(1-\omega_1\zeta_1)^{-1}\omega_1\zeta_1 = (1-\omega_1\zeta_1)^{-1},\end{aligned}$$

and making a change of odd coordinates $\zeta'\mapsto(1+N)\zeta'$, the Berezin integral over $D(\zeta',\omega')$ simplifies to

$$(1-\omega_1\zeta_1)^{q-1}\int D(\zeta',\omega')\,\Delta_{\mathbf{m}''+(q-p)\mathbb{1}''}\begin{pmatrix}1-\omega_1\zeta_1 & 0\\ 0 & 1-\omega'\zeta'\end{pmatrix}^{-1}.$$

We obtain

$$\begin{aligned}\gamma_{-(\mathbf{m}''+(q-p)\mathbb{1}'')} &= \gamma_{-m_{p+1}-1+p}\gamma_{-(m_{p+2}+q-p,\ldots,m_{p+q}+q-p)}\\ &= \cdots = \prod_{j=1}^{q}\gamma_{-m_{p+j}-j+p}.\end{aligned}$$

Finally, writing $a \coloneqq 1-\sum_{j=2}^{p}\omega_{1j}\zeta_{j1}$, we compute

$$(1-\omega_1\zeta_1)^{-m} = a^{-m}(1-a^{-1}\omega_{11}\zeta_{11})^{-m} = a^{-m-1}(1+m\omega_{11}\zeta_{11})$$

which recursively gives

$$\gamma_{-m} = m(m+1)\cdots(m+p-1) = \frac{\Gamma(m+p)}{\Gamma(m)},$$

and hence, our claim. □

Lemma 3.21. *We have*

$$\int_{\mathrm{U}(q)} |Dw| e^{\operatorname{tr}(w)} (\Delta_{\mathbf{m}''}(w))^{-1} = \prod_{k=1}^{q} \frac{\Gamma(q-(k-1))}{\Gamma(m_{p+k}+q-(k-1))},$$

where $\mathbf{m}'' := (m_{p+1}, \ldots, m_{p+q})$.

Proof. We use spherical polynomials for this computation. They are defined as

$$\Phi_{\mathbf{r}}(x) := \int_{\mathrm{U}(q)} |Dw| \, \Delta_{\mathbf{r}}(wx),$$

where $\mathbf{r}$ is an arbitrary multi-index of length q.

Thanks to [9, Proposition XII.1.3.(i)] we can write the exponential in this integral as an absolutely convergent series of spherical functions:

$$e^{\operatorname{tr}(w)} = \sum_{\mathbf{n} \geqslant 0} \frac{d_{\mathbf{n}}}{q_{\mathbf{n}}} \Phi_{\mathbf{n}}(w),$$

where $d_{\mathbf{n}}$ is the dimension of the finite-dimensional irreducible $\mathrm{U}(q)$-module of highest weight $\mathbf{n}$, and

$$q_{\mathbf{n}} := \prod_{k=1}^{q} \frac{\Gamma(n_k+q-(k-1))}{\Gamma(q-(k-1))}.$$

Therefore, we can write our integral as

$$\sum_{\mathbf{n} \geqslant 0} \frac{d_{\mathbf{n}}}{q_{\mathbf{n}}} \int_{\mathrm{U}(q)} |Dw| \, \Phi_{\mathbf{n}}(w) (\Delta_{\mathbf{m}''}(w))^{-1}.$$

Notice that since $w \in \mathrm{U}(q)$, we have $(\Delta_{\mathbf{m}''}(w))^{-1} = \Delta_{\mathbf{m}''}(w^{-1})$. Due to the $\mathrm{U}(q)$-invariance of $\Phi_{\mathbf{n}}$, we have

$$\int_{\mathrm{U}(q)} |Du| \, \Phi_{\mathbf{n}}(xu) = \Phi_{\mathbf{n}}(x),$$

so we obtain

$$\begin{aligned}
\int_{\mathrm{U}(q)} |Dw| \, \Phi_{\mathbf{n}}(w) \Delta_{\mathbf{m}''}(w^{-1}) &= \int_{\mathrm{U}(q)\times\mathrm{U}(q)} |Dw||Du| \, \Phi_{\mathbf{n}}(wu) \Delta_{\mathbf{m}''}(w^{-1}) \\
&= \int_{\mathrm{U}(q)\times\mathrm{U}(q)} |Dw||Du| \, \Phi_{\mathbf{n}}(w) \Delta_{\mathbf{m}''}(uw^{-1}) \\
&= \int_{\mathrm{U}(q)} |Dw| \, \Phi_{\mathbf{n}}(w) \Phi_{\mathbf{m}''}(w^{-1}).
\end{aligned}$$

Further, it is easy to see that $\Phi_{\mathbf{m}''}(w^{-1}) = \Phi_{\mathbf{m}''}(w^*) = \overline{\Phi_{\mathbf{m}''}(w)}$.

By the classical Schur orthogonality relations, this integral is only non-zero when $\mathbf{m}'' = \mathbf{n}$, in which case the answer is $1/d_{\mathbf{m}''}$. Therefore,

$$\int_{\mathrm{U}(q)} |Dw| \, e^{\operatorname{tr}(w)} (\Delta_{\mathbf{m}''}(w))^{-1} = 1/q_{\mathbf{m}''},$$

which gives the claim. ∎

Appendix A. The functor of points

A.1. **The language of S-valued points.** For a manifold X, a point can be thought of as a morphism $* \to X$, and this completely determines X. However, for a supermanifold X, such a morphism $* \to X$ is again only a point of the underlying manifold X_0, and therefore does not capture the supergeometric features of X. To deal with this, the notion of points has to be extended.

This idea is familiar in algebraic geometry. Here, it is common to talk about the K-rational points of a scheme X, which are nothing but morphisms $\operatorname{Spec}(K) \to X$.

Grothendieck extended this idea and considered the scheme along with its A-points, for all commutative rings A. Then X is completely recaptured by its collection of A-points, for all commutative rings A, along with admissible morphisms.

More generally, if $\mathbf{C}$ is any category, and X is an object of $\mathbf{C}$, then an *S-valued point* (where S is another object of $\mathbf{C}$) is defined to be a morphism $x : S \to X$. One may view this as a 'deformed' or 'parametrised' point. Suggestively, one writes $x \in_S X$ in this case, and denotes the set of all $x \in_S X$ by $X(S)$.

For any morphism $f : X \to Y$, one may define a set-map $f_S : X(S) \to Y(S)$ by

$$f_S(x) \coloneqq f(x) \coloneqq f \circ x \in_S Y \quad \text{for all} \quad x \in_S X.$$

Clearly, the values $f(x)$ completely determine f, as can be seen by evaluating at the *generic point* $x = \mathrm{id}_X \in_X X$.

In fact, more is true. The following statement is known as Yoneda's Lemma [21]: Given a collection of set-maps $f_S : X(S) \to Y(S)$, there exists a morphism $f : X \to Y$ such that $f_S(x) = f(x)$ for all $x \in_S X$ if and only if

$$f_T(x(t)) = f_S(x)(t) \quad \text{for all} \quad t : T \to S.$$

The points $x(t)$ are called *specialisations of x*, so the condition states that the collection (f_S) is invariant under specialisation.

The above facts are usually stated in the following more abstract form: For any object X, we have a set-valued functor $X(-) : \mathbf{C}^{op} \to \mathbf{Sets}$, and the set of natural transformations $X(-) \to Y(-)$ is naturally bijective to the set of morphisms $X \to Y$. Thus, the functor $X \mapsto X(-)$ from $\mathbf{C}$ to $[\mathbf{C}^{op}, \mathbf{Sets}]$, called the *Yoneda embedding*, is fully faithful.

The Yoneda embedding preserves products [21], so if $\mathbf{C}$ admits finite products, it induces a fully faithful embedding of the category of group objects in $\mathbf{C}$ into the category $[\mathbf{C}^{op}, \mathbf{Grp}]$ of group-valued functors.

In other words, we have the following: Let X be an object in $\mathbf{C}$. Then X is a group object if and only if for any S, $X(S)$ admits a group law, which is invariant under specialisation. We will assiduously apply this point of view to the categories of complex supermanifolds and of *cs* manifolds.

A.2. **Vector fields and generalised points.** We now show how vector fields can be understood in terms of generalised points. Among other things, this is a framework enabling us to exchange super-integration and differentiation under suitable assumptions.

Let X and S be *cs* manifolds. Then X is called a cs *manifold over S*, written X/S, if supplied with some morphism $X \to S$, which on some open cover U_α of X fits into a commutative diagram

$$\begin{array}{ccc} U_\alpha & \longrightarrow & S \times Y \\ \big\downarrow & & \big\downarrow{\scriptstyle p_1} \\ V_\alpha & \longrightarrow & S \end{array}$$

where the rows are open embeddings. Usually, we will consider only products, but the general language will be efficient nonetheless. There is an obvious notion of morphisms over S, which we denote $X/S \to Y/S$.

A system of (local) *fibre coordinates* is given by the system $(x^a) = (x, \xi)$ of superfunctions on some trivialising open subspace $U \subseteq X$ obtained by pullback along a trivialisation from a coordinate system in the fibre Y.

If X/S is a *cs* manifold over S, then the *relative tangent sheaf* is defined by

$$\mathcal{T}_{X/S} \coloneqq \underline{\mathcal{D}er}_{p_0^{-1}\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X),$$

the sheaf of superderivations of $\mathcal{O}_X$ which are linear over $\mathcal{O}_S$. Here, p denotes the morphism $X \to S$. It is a basic fact that $\mathcal{T}_{X/S}$ is a locally free $\mathcal{O}_X$-module, with rank equal to the fibre dimension of X/S.

More generally, let $\varphi : X/S \to Y/S$ be a morphisms over S. We let

$$\mathcal{T}_{X/S\to Y/S} := \underline{\mathcal{D}er}_{p_0^{-1}\mathcal{O}_S}(\varphi_0^{-1}\mathcal{O}_Y, \mathcal{O}_X)$$

and call this the *tangent sheaf along* φ *over* S. Written out explicitly, the derivation property of a homogeneous element $\delta \in \mathcal{T}_{X/S\to Y/S}(U)$ is

$$\delta(fg) = \delta(f)\varphi^\sharp(g) + (-1)^{|f||\delta|}\varphi^\sharp(f)\delta(g)$$

for all homogeneous $f, g \in \mathcal{O}_Y(V)$, $V \subseteq Y_0$ an open neighbourhood of $\varphi_0(U)$. The usual relative tangent bundle corresponds to $\varphi = \mathrm{id}_X$.

We denote by $S[\varepsilon, \tau]$ the $\mathbb{C}$-superspace $(S_0, \mathcal{O}_S[\varepsilon, \tau])$, where $\mathcal{O}_S[\varepsilon, \tau] := \mathcal{O}_S \otimes_{\mathbb{C}} A$, A denoting the superalgebra $\mathbb{C}[\varepsilon, \tau]/(\varepsilon^2, \varepsilon\tau)$, where ε and τ are understood to be even and odd indeterminates, respectively. If X/S and Y/S are *cs* manifolds over S and $\varphi : X/S \to Y/S$ is a morphism over S, then there is a natural bijection

$$\big\{\gamma \in \mathrm{Hom}_S\big(X[\varepsilon, \tau], Y\big) \bigm| \gamma|_{\varepsilon=\tau=0} = \varphi\big\} \to \Gamma(\mathcal{T}_{X/S\to Y/S}) : \gamma \mapsto \delta$$

given by the equation $\gamma^\sharp(f) = \varphi^\sharp(f) + \varepsilon\delta_{\bar 0}(f) + \tau\delta_{\bar 1}(f)$ for all local sections f of $\mathcal{O}_Y$.

In particular, consider the case of a *cs* Lie supergroup G. The Lie superalgebra $\mathfrak{g}$ is by definition the fibre over 1 of the tangent bundle. Equivalently, elements of $\mathfrak{g}$ may be seen as vector fields along the morphism $1_G : 1 \to G$, that is, as $*[\tau, \varepsilon]$-valued points of G along 1_G.

A.3. **Nilpotent shifts of cycles in middle dimension.** We will now show how the technique of nilpotent shifts common in physics can be understood in terms of S-valued points.

Let $W \cong \mathbb{C}^q$ be a complex vector space. Its associated $2q$-dimensional real manifold is naturally a *cs* manifold with sheaf $\mathcal{O}_W$ of smooth complex-valued functions on $\mathbb{R}^{2q} \cong W$.

Abusing notation, we write N for the *cs* manifold $\mathbb{R}^{0|N}_{\mathbb{C}}$. Assume that

$$n \in \Gamma(\mathcal{N}_{N,\bar 0})^{2q} = \Gamma(\mathcal{N}_{N,\bar 0} \otimes_{\mathbb{C}} W_{\mathbb{C}}),$$

where $\mathcal{N}_N = \bigwedge^+(\mathbb{C}^N)^*$ is the ideal of $\mathcal{O}_N = \bigwedge(\mathbb{C}^N)^*$ generated by $\mathcal{O}_{N,\bar 1}$.

The generic point $w = \mathrm{id}_W \in_W W$ of W corresponds by Leites's Theorem [19] to the element $w = \sum_i e^i \otimes e_i \in \Gamma(\mathcal{O}_W \otimes_{\mathbb{C}} W_{\mathbb{C}})$ where e_i, $i = 1, \dots, 2q$, is a real basis of W, and e^i is its dual basis. The sum $w + n \in \Gamma(\mathcal{O}_W \otimes_{\mathbb{C}} W_{\mathbb{C}})$ corresponds by Leites's Theorem to a unique morphism $\phi : W \times N \to W$.

In particular, this gives a definite meaning to $f(w + n) = \phi^\sharp(f)$ for any smooth complex-valued function f defined on an open subset of W. It is known that

$$f(w + n) = f(w) + \sum_{k=1}^{N} \frac{1}{k!} d^k f(w)(n, \dots, n)$$

where the derivatives are extended multi-linearly over $\mathcal{N}_{N,\bar 0}$.

Let now X be a closed real submanifold of W of dimension q. We call such an X a *mid-dimensional cycle.* Assume that

$$n \in \Gamma(\mathcal{N}_{N,\bar 0})^q = \Gamma(\mathcal{N}_{N,\bar 0} \otimes_{\mathbb{C}} W).$$

In this case, we call n a *nilpotent shift.*

By the use of the embedding $j : X \to W$, the real tangent space at any point of X is naturally identified with a q-dimensional real subspace of W, and this gives a vector bundle map $Tj : TX \to X \times W$. Thus, the complex tangent space at any

point is naturally identified with W, which gives an isomorphism of complex vector bundles $T^{\mathbb{C}}j : T^{\mathbb{C}}X \to X \times W$.

For any $f \in \mathcal{O}_X(U)$, where $U \subseteq X$ is open, and any $y \in_S U$, we define

$$f(y+n) \coloneqq f(y) + \sum_{k=1}^{N} \frac{1}{k!} T^k f(y)((T^{\mathbb{C}}j)^{-1}(y,n), \dots, (T^{\mathbb{C}}j)^{-1}(y,n))$$

by multi-linear extension of the higher order tangent maps. Here, the left hand side lies in $\Gamma(\mathcal{O}_{S\times N})$. In particular, this defines a unique morphism $X \times N \to X$, which sends f to $f(x+n)$, $x = \mathrm{id}_X \in_X X$ denoting the generic point of X.

Appendix B. Integration on supermanifolds

We will need to consider super-integrals depending on some parameters. The correct framework for this is that of relative Berezinians, paired with the understanding of parameter dependence in terms of S-valued points.

B.1. **Relative Berezinians and fibre integrals.** Let X be a *cs* manifold over S, and $\Omega^1_{X/S}$ be the module of relative 1-forms, by definition dual to $\mathcal{T}_{X/S}$. Then we define the sheaf of *relative Berezinians* $\mathcal{B}er_{X/S}$ to be the Berezinian sheaf associated to the locally free $\mathcal{O}_X$-module $\Pi\Omega^1_{X/S}$ obtained by parity reversal. Furthermore, the sheaf of *relative Berezinian densities* $|\mathcal{B}er|_{X/S}$ is the twist by the relative orientation sheaf, *i.e.*

$$|\mathcal{B}er|_{X/S} \coloneqq \mathcal{B}er_{X/S} \otimes_{\mathbb{Z}} or_{X_0/S_0}.$$

Given a system of local fibre coordinates $(x^a) = (x,\xi)$ on U, their coordinate derivations $\frac{\partial}{\partial x^a}$ form an $\mathcal{O}_X|_{U_0}$-module basis of $\mathcal{T}_{X/S}|_{U_0}$, with dual basis dx^a of $\Omega^1_{X/S}|_{U_0}$. One may thus consider the distinguished basis

$$|D(x^a)| = |D(x,\xi)| = dx_1 \dots dx_p \frac{\partial^{\Pi}}{\partial \xi^1} \cdots \frac{\partial^{\Pi}}{\partial \xi^q}$$

of the module of Berezinian densities $|\mathcal{B}er|_{X/S}$, *cf.* [22].

If X/S is a direct product $X = S \times Y$, then

$$|\mathcal{B}er|_{X/S} = p_2^*\big(|\mathcal{B}er|_Y\big) = \mathcal{O}_X \otimes_{p_{2,0}^{-1}\mathcal{O}_Y} p_{2,0}^{-1}|\mathcal{B}er|_Y.$$

In particular, the integral over Y of compactly supported Berezinian densities defines the integral over X of a section of $(p_0)_!|\mathcal{B}er|_{X/S}$, where $(-)_!$ denotes the functor of direct image with compact supports [17]. We denote the quantity thus obtained by

$$\fint_{S}\!\!\int_X \omega \in \Gamma(\mathcal{O}_S) \quad \text{for all } \ \omega \in \Gamma\big((p_0)_!|\mathcal{B}er|_{X/S}\big),$$

and call this the *fibre integral* of ω.

We will, however, have to consider fibre integrals in a more general setting, beyond compact supports. Henceforth, we assume for simplicity that $X = S \times Y$. A *fibre retraction* for X is a morphism $r : Y \to Y_0$ which is left inverse to the canonical embedding $j : Y_0 \to Y$, where Y_0 denotes the underlying manifold of Y.

A system of fibre coordinates (x,ξ) of X/S is called *adapted* to r if $x = r^\sharp(x_0)$. Given an adapted system of fibre coordinates, we may write $\omega = |D(x,\xi)|\, f$ and

$$f = \sum_{I\subseteq\{1,\dots,q\}} (\mathrm{id}\times r)^\sharp(f_I)\, \xi^I$$

for unique coefficients $f_I \in \Gamma(\mathcal{O}_{S\times Y_0})$, where $\dim Y = *|q$. Then one defines

$$\fint_{S\times Y_0}\!\!\int_X \omega \coloneqq |dx_0|\, f_{\{1,\dots,q\}} \in \Gamma(|\mathcal{B}er|_{(S\times Y_0)/S}).$$

Note that $|\mathcal{B}er|_{(S\times Y_0)/S}$ is p_2^* of the sheaf of ordinary densities on the manifold Y_0, so we may write $|dx_0|$.

This fibre integral only depends on r, and not on the choice of an adapted system of fibre coordinates. If the resulting relative density is absolutely integrable along the fibre Y_0, then we say that ω is *absolutely integrable* with respect to r, and define

$$ {}_S\!\!\fint_X \omega := {}_S\!\!\fint_{S\times Y_0} \left[{}_{S\times Y_0}\!\!\fint_X \omega \right] \in \Gamma(\mathcal{O}_S). $$

Both this quantity and its existence depend heavily on r.

Using the topology on $\Gamma(\mathcal{O}_S)$ introduced below, in Appendix C, the convergence may be understood in terms of vector-valued integrals. The relative density ${}_{S\times Y_0}\!\fint_X \omega$ may be viewed as a $\Gamma(\mathcal{O}_S)$-valued density on Y_0. It is absolutely integrable along Y_0 if and only if the corresponding vector-valued density is Bochner integrable.

We shall use the language of S-points discussed above in Appendix A.1 to manipulate integrals of relative Berezinian densities in a hopefully more comprehensible formalism. This also gives a rigorous foundation for the super-integral notation common in the physics literature.

If f is a superfunction on $X = S \times Y$ and we are given some relative Berezinian density $|Dy|$ on X/S, then we write

$$ \int_Y |Dy|\, f(s,y) := {}_S\!\!\fint_X |Dy|\, f. $$

This is justified by the convention that the generic points of S and Y are denoted by s and y, respectively. Moreover, it is easy to see that this notation behaves well under specialisation, since

$$ \int_Y |Dy|\, f(s(t),y) = {}_T\!\!\fint_X (t\times \mathrm{id})^\sharp(|Dy|\, f) = t^\sharp \left[{}_S\!\!\fint_X |Dy|\, f \right] \tag{B.1} $$

for any $t \in_S S$. This follows from the fact that the fibre retractions are respected by the morphism $s \times \mathrm{id}$.

B.2. **Berezin integrals and nilpotent shifts.** We now return to the nilpotent shifts previously considered in Appendix A.3, and apply them to certain integrals.

Let X be a mid-dimensional cycle in the complex vector space $W \cong \mathbb{C}^q$. Let X carry some pseudo-Riemannian metric g and μ_g be the induced Riemannian density.

The following is a straightforward generalisation of [20, Lemma 4.13].

Lemma B.1. *Let n be a nilpotent shift for X. Then for any compactly supported smooth function f on X, we have*

$$ \int_X f(x+n)\, d\mu_g(x) = \int_X f(x) J_t(x)\, d\mu_g(x), $$

where $J_t \in \Gamma(\mathcal{O}_{X\times N})$ is the solution of the ODE

$$ \frac{d}{dt} \log J_t(y,s) = -\operatorname{div} v_n(y - tn, s) $$

with inital condition $J_0 = 1$ and v_n is the vector field on $X \times N$ over N, defined by

$$ v_n(h)(y,s) := \frac{d}{dt} h(y+tn, s)\Big|_{t=0} $$

for any smooth h on some open subspace $U \subseteq X \times N$ and any $(y,s) \in_S X \times N$.

Proof. Consider for fixed $t \in \mathbb{R}$ the morphism $\phi_t : X \times N \to X \times N$, defined by

$$\phi_t(y,z) \coloneqq (y - tn, z)$$

for any $(y,z) \in_S X \times N$.

Then ϕ_t is an isomorphism over N, with inverse ϕ_{-t}. We may consider μ_g as a Berezinian density of $X \times N$ over N. Thus, we have

$$\int_X f(x+tn)\,d\mu_g(x) = {}_N\!\fint_{X\times N} \phi_{-t}^\sharp(p_1^\sharp(f))\,\mu_g = {}_N\!\fint_{X\times N} p_1^\sharp(f)\,\phi_t^\sharp(\mu_g).$$

Since $\mathcal{B}er_{X\times N/N}$ is a free $\mathcal{O}_{X\times N}$-module with module basis μ_g, there exists a unique even $J_t \in \Gamma(\mathcal{O}_{X\times N})$ such that

$$\phi_t^\sharp(\mu_g) = J_t\,\mu_g.$$

These superfunctions depend smoothly on t. Indeed, one may consider the morphism $\phi : \mathbb{R} \times X \times N \to \mathbb{R} \times X \times N$, given by

$$\phi(t,y,z) \coloneqq (t, y - tn, z)$$

for all $(t,y,z) \in_S \mathbb{R}\times X\times N$. Then ϕ is an isomorphism over $\mathbb{R}\times N$, and $\phi(t,y,z) = (t, \phi_t(y,z))$ if t is the specialisation of an ordinary point of $\mathbb{R}$.

Now, ϕ_t is the flow of the vector field $-v_n$, considered as vector field on $X\times N/N$. Indeed, if h is a smooth function on $\mathbb{R}\times X\times N$, then

$$\frac{d}{dt}\phi^\sharp(p_{23}^\sharp(h))(t,y,z) = \frac{d}{ds}h(y-(s+t)n,z)|_{s=0} = -\phi^\sharp(p_{23}^\sharp(v_n h))(t,y,z).$$

We have

$$\frac{d}{dt}J_t = \frac{d}{d\tau}\phi_t^\sharp(J_\tau)\Big|_{\tau=0} = -\phi_t^\sharp(\operatorname{div}_g v_n)\cdot J_t,$$

so

$$\frac{d}{dt}\log J_t(y,s) = -\phi_t^\sharp(\operatorname{div} v_n)(y,s) = -\operatorname{div} v_n(y-tn,s)$$

with $J_0 = 1$, proving the lemma. □

Let G be a Lie group acting linearly on W and $o \in W$ such that $K \coloneqq G_o$ is a compact subgroup, open in the fixed point set of an involutive automorphism θ of G. Then the orbit $X \coloneqq G.o = G/K$ is a Riemannian symmetric space.

Also denote by θ the involutive automorphism of the Lie algebra $\mathfrak{g}$ of G induced by θ. Then $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ where $\mathfrak{k}$, the $+1$ eigenspace of θ, is the Lie algebra of K, and $\mathfrak{p}$, the -1 eigenspace of θ, identifies as a K-module with $T_oX = \mathfrak{g}/\mathfrak{k}$.

We will denote the action of G on W by $g.w$, and the derived action of $\mathfrak{g}$ by $u.w$. Since $\mathfrak{p}_{\mathbb{C}} = W$, we obtain for any $w \in W$ endomorphisms R_w of $\mathfrak{p}_{\mathbb{C}}$ by $R_w(u) \coloneqq u.w$.

In this setting, we have the following generalisation of [20, Lemma 4.12].

Lemma B.2. *Assume that $X = G/K$ is a mid-dimensional cycle. Then*

$$(\operatorname{div} v_n)(gK) = -\tfrac{1}{2}\operatorname{tr}_{\mathfrak{p}_{\mathbb{C}}}(R_{g^{-1}.n})$$

for any nilpotent shift n.

Proof. Let ∇ denote the Levi-Civita connection of X. For any vector field v on X, one has the identity

$$\operatorname{div} v = \operatorname{tr}\nabla v$$

where ∇v denotes the endomorphism of the tangent sheaf $\mathcal{T}_X$ given by $u \mapsto \nabla_u v$. This statement extends immediately to the case of relative vector fields.

It is known [27, Proposition 5.2] that the space of G-invariant connections on X is in bijection with the set of all K-equivariant linear maps

$$\lambda : \mathfrak{g} \to \operatorname{End}(\mathfrak{p})$$

whose restriction to $\mathfrak{k}$ equals the adjoint action. Since $[\mathfrak{p},\mathfrak{p}]\subseteq\mathfrak{p}$, the map λ corresponding to ∇ is given by

$$\lambda(x+y)\coloneqq \operatorname{ad}x \quad \text{for all}\quad x\in\mathfrak{k}, y\in\mathfrak{p},$$

in view of [27, Proposition 5.9]; identifying TX with $G\times^K\mathfrak{p}$, we have

$$(\nabla_{[g,u]}v)_{gK}f = ([u_X,v]f)(gK) - [g,\lambda(\operatorname{Ad}(g^{-1})(u))(T_oL_g)^{-1}v_{gK}]f$$

for $[g,u]\in T_{gK}X$, $v\in\Gamma(\mathcal{T}_X)$, and any smooth function f on an open neighbourhood of gK. Here, u_X denotes the vector field defined by $u\in\mathfrak{p}\subseteq\mathfrak{g}$ *via* the derived action on X, and L_g is the action of g on X.

Explicitly, the identification of $G\times^K\mathfrak{p}$ with TX maps $[g,u]$ to $\dot\gamma(0)$ where $\gamma(t)=g\exp(tu)K$. Therefore, the value at gK of the G-invariant vector field $\vec{u}$ defined by $u\in\mathfrak{p}$ is just given by $[g,u]$.

By the invariance of the trace, we have

$$\operatorname{tr}_{T_{gK}X}(\nabla v) = \operatorname{tr}_{T_oX}\big([1,u]\mapsto T_{gK}L_{g^{-1}}(\nabla_{[g,u]}v)_{gK}\big).$$

Denoting the action of G on vector fields by $g\cdot v$, we compute, using the G-invariance of the Levi-Civita connection, that

$$T_{gK}L_{g^{-1}}(\nabla_{[g,u]}v)_{gK} = (g^{-1}\cdot\nabla_{\vec{u}}v)_o = (\nabla_{g^{-1}\cdot\vec{u}}(g^{-1}\cdot v))_o = (\nabla_{[1,u]}(g^{-1}\cdot v))_o.$$

In particular, since $\lambda(u)=0$ for all $u\in\mathfrak{p}$, we find

$$(\operatorname{div}v)(gK) = \operatorname{tr}_{\mathfrak{p}}\big(u\mapsto[u_X,g^{-1}\cdot v]_o\big)$$

for any vector field v. We may replace $\operatorname{tr}_{\mathfrak{p}}$ by $\frac12\operatorname{tr}_{\mathfrak{p}_{\mathbb{C}}}$.

Using the identification $T^{\mathbb{C}}j$ of $X\times W$ with $T^{\mathbb{C}}X = G\times^K\mathfrak{p}_{\mathbb{C}}$ given by the G-action on W, we see $(v_n)_{gK}=[g,g^{-1}.n]$. Hence, one finds that

$$\begin{aligned}(g^{-1}\cdot v_n)_{hK} &= T_{ghK}L_{g^{-1}}(v_n)_{ghK} = T_{ghK}L_{g^{-1}}[gh,(gh)^{-1}.n]\\ &= [h,h^{-1}.g^{-1}.n] = (v_{g^{-1}.n})_{hK}.\end{aligned}$$

Computing the commutator

$$[u_X,v_n] = \frac{d}{ds}\exp(-su)\cdot v_n\Big|_{s=0} = \frac{d}{ds}v_{\exp(-su).n}\Big|_{s=0} = -v_{u.n},$$

we see that

$$(\operatorname{div}v_n)(gK) = \tfrac12\operatorname{tr}_{\mathfrak{p}_{\mathbb{C}}}\big(u\mapsto[u_X,v_{g^{-1}.n}]_o\big) = -\tfrac12\operatorname{tr}_{\mathfrak{p}_{\mathbb{C}}}R_{g^{-1}.n},$$

thus completing the proof of the lemma. □

Appendix C. Superdistributions and Laplace transforms

In this appendix, we develop some basic Euclidean Fourier analysis for superdistributions. The facts about Fourier inversion on the Schwartz space are well-known, but we are not aware of a convenient reference. The account we give of the Laplace transform is to our knowledge new.

C.1. **Superdistributions.** In this subsection, we give a self-contained development of the basic functional analytic properties of the spaces of superfunctions and superdistributions we encounter in this article.

All the results can also be derived quickly from the classical case by invoking Batchelor's theorem. However, we deliberately avoid this point of view on grounds that it is generally useful to have definitions of the relevant topologies at hand, which do not appeal to coordinates in their definition.

We use some well-established functional analysis terminology at liberty. Basic texts include Refs. [23,30,34].

Definition C.1 (Temperedness and Schwartz class)**.** Let $V_{\bar 0,\mathbb{R}}$ be a real vector space and $\|\cdot\|$ a norm function on $V_{\bar 0,\mathbb{R}}$. A complex-valued function f on $V_{\bar 0,\mathbb{R}}$ is of *moderate growth* if there exists $N \in \mathbb{N}$ such that $|f(x)| \leqslant \|x\|^N$ for all $x \in V_{\bar 0,\mathbb{R}}$.

Let $(V, V_{\bar 0,\mathbb{R}})$ be a *cs* vector space. Consider the associated *cs* manifold $L(V, V_{\bar 0,\mathbb{R}})$, with sheaf of superfunctions $\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}}$.

For $f \in \Gamma(\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}})$ and $D \in S(V)$, considered as a differential operator, we write

$$f(D; x) \coloneqq (Df)(x)$$

for all $x \in V_{\bar 0,\mathbb{R}}$. Then f is called *tempered* if for any $D \in S(V)$, the function $f(D; \cdot)$ is of moderate growth; it is of *Schwartz class* if for any $D \in S(V)$, and any tempered superfunction h, the function $(hf)(D; \cdot)$ is bounded. In the latter case, we set

$$p_{h,D}(f) \coloneqq \sup\nolimits_{x \in V_{\bar 0,\mathbb{R}}} |(hf)(D; x)|.$$

Then f is tempered if any only if for every $D \in S(V)$, we have

$$\sup\nolimits_{x \in V_{\bar 0,\mathbb{R}}} \|x\|^{-N} |f(D; x)| < \infty$$

for some $N > 0$; f is of Schwartz class if and only if for $D \in S(V)$, we have

$$p_{N,D}(f) \coloneqq \sup\nolimits_{x \in V_{\bar 0,\mathbb{R}}} \|x\|^{N} |f(D; x)| < \infty$$

for all $N > 0$. The totality of all tempered superfunctions (resp. superfunctions of Schwartz class) is denoted by $\mathscr{T}(V)$ (resp. $\mathscr{S}(V, V_{\bar 0,\mathbb{R}})$). We endow $\mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ with the locally convex topology defined by the seminorms $p_{h,D}$ (or, equivalently, $p_{N,D}$), and let $\mathscr{S}'(V, V_{\bar 0,\mathbb{R}})$ denote the topological dual space of $\mathscr{S}(V, V_{\bar 0,\mathbb{R}})$, with the strong topology. The elements of $\mathscr{S}'(V, V_{\bar 0,\mathbb{R}})$ are called *tempered superdistributions.*

Lemma C.2. *The space $\mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ is a Fréchet space and in particular, barrelled.*

Proof. In view of the above discussion, $\mathscr{S}(V, V_{\bar 0,\mathbb{R}}) \cong \mathscr{S}(V_{\bar 0,\mathbb{R}})^N$ where $N = 2^{\dim V_{\bar 1}}$. The assertion follows from [12, Chapter II, Section 2.2, Theorem 2]. ☐

If X is a *cs* manifold, then we endow $\mathcal{O}_X(U)$, for any open set $U \subseteq X_0$, with the locally convex topology induced by the seminorms

$$p_{K,D}(f) \coloneqq \sup_{x \in K} \big|(Df)(x)\big|,$$

where D runs throught the set $\mathcal{D}_X(U)$ of superdifferential operators (of finite order) on X_U, and $K \subseteq U$ is compact. In what follows, we require X_0 to be *metrisable* (or equivalently, paracompact [29, Appendix]).

Proposition C.3. *Let (U_α) be an open cover of U. Then $\mathcal{O}_X(U)$ is the locally convex projective limit of the $\mathcal{O}_X(U_\alpha)$, with respect to the restriction morphisms. In particular, $\mathcal{O}_X(U)$ is complete, and if U is σ-compact, then $\mathcal{O}_X(U)$ is Fréchet.*

Proof. Since $\mathcal{D}_X$ is an $\mathcal{O}_X$-module and $\mathcal{O}_X$ is c-soft, so is $\mathcal{D}_X$. This readily implies that the restriction maps are continuous. Hence, we have that the linear map $\mathcal{O}_X(U) \to \varprojlim_\alpha \mathcal{O}_X(U_\alpha)$ is continuous, and is bijective by the sheaf property. Conversely, to see that it is open, one may pass to a locally finite refinement, and then argue similarly using partitions of unity. The remaining statements then carry over from the case of coordinate neighbourhoods, which is easily dealt with. ☐

Remark C.4. If X is a complex supermanifold, then the sheaf $\mathcal{O}_X$ of (holomorphic) superfunctions embeds to into the sheaf $\mathcal{O}_{X_{cs}}$ of (smooth) superfunctions on the associated *cs* manifold X_{cs}. As follows from the case of an open subspace of $\mathbb{C}^p$, $\mathcal{O}_X(U)$, endowed with the relative topology induced from $\mathcal{O}_{X_{cs}}(U)$, is a (nuclear) Fréchet space.

Moreover, if X is an open subspace of the supermanifold $L(V)$ associated with a complex super-vector space V, then the topology on $\mathcal{O}_X(U)$ is generated by the seminorms $p_{K,D}$, where $K \subseteq U$ runs through the compact subsets, and $D \in \bigwedge(V_{\bar 1})$.

We recall that if A is an algebra (not necessarily unital or associative) endowed with a locally convex topology, then this topology is called *locally m-convex* if it is generated by a system of submultiplicative seminorms.

Corollary C.5. *The topology on $\mathcal{O}_X(U)$ is locally m-convex.*

Proof. In view of Proposition C.3, it is sufficient to prove this in a coordinate neighbourhood. Then, as in the even case [24, 2.2], a locally m-convex topology is generated by the seminorms

$$p_{K,k}(f) = 2^k \cdot \max_{\alpha\in\mathbb{N}^p\times\{0,1\}^q|\alpha|\leqslant k} \sup_{x\in K} |f(\partial^\alpha, x)|,$$

for $k \in \mathbb{N}$ and $K \subseteq U$ compact, where we agree to write

$$\partial^\alpha := \frac{\partial^{\alpha_{p+q}}}{\partial(x^{p+q})^{\alpha_{p+q}}} \cdots \frac{\partial^{\alpha_1}}{\partial(x^{p+q})^{\alpha_1}},$$

and (x^a) is some local coordinate system on U. □

By similar arguments as the proof of Proposition C.3, one proves the following.

Proposition C.6. *Let $\phi : X \to Y$ be a morphism of* cs *manifolds. Then the even linear pullback map $\phi^\sharp : \Gamma(\mathcal{O}_Y) \to \Gamma(\mathcal{O}_X)$ is continuous.*

Let X be a *cs* manifold where X_0 is σ-compact.[3] For any compact $K \subseteq X_0$, let $\Gamma_K(\mathcal{O}_X)$ denote the set of all global sections of $\mathcal{O}_X$ with support in K. Endowed with the relative topology from $\Gamma(\mathcal{O}_X) = \mathcal{O}_X(X_0)$, it is a Fréchet space.

Let $\Gamma_c(\mathcal{O}_X) = \bigcup_K \Gamma_K(\mathcal{O}_X)$, where the union extends over all compact subsets $K \subseteq X_0$, be the set of all compactly supported sections of $\mathcal{O}_X$, equipped with the locally convex inductive limit topology.

Proposition C.7. *The locally convex space $\Gamma_c(\mathcal{O}_X)$ has the following properties:*

(i). *It is LF, and in particular, complete, barrelled and bornological.*
(ii). *It is nuclear, and in particular, reflexive and Montel.*

The latter statement also holds for $\Gamma(\mathcal{O}_X)$.

Proof. (i). If $K' \supseteq K$, then $\Gamma_K(\mathcal{O}_X) \to \Gamma_{K'}(\mathcal{O}_X)$ is by definition a topological embedding. Since X_0 is σ-compact, the limit topology is computed by taking any countable exhaustive filtration of X_0 by compact subsets.

(ii). It is sufficient to prove the nuclearity of $\Gamma_K(\mathcal{O}_X)$, since this property is preserved under countable locally convex inductive limits [34, Proposition 50.1]. The same holds true for locally convex projective limits (*loc. cit.*), so the question is reduced to the case of *cs* domain, in view of Proposition C.3. In this case, $\Gamma_K(\mathcal{O}_X) \cong \mathcal{C}^\infty_K(U)^N$ where $U \subseteq \mathbb{R}^p$ is open, $K \subseteq U$ is compact, and $N = 2^q$ is some non-negative integer. The claim then follows from *loc. cit.* and the Corollary to Theorem 51.4 (*op. cit.*). For $\Gamma(\mathcal{O}_X)$, we argue analogously. □

Corollary C.8. *The locally convex spaces $\mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ and $\mathscr{S}'(V, V_{\bar 0,\mathbb{R}})$ are nuclear, barrelled, reflexive, and Montel.*

The *proof* makes use of the following lemma.

Lemma C.9. *The natural maps $\Gamma_c(\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}}) \to \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ and $\mathscr{S}'(V, V_{\bar 0,\mathbb{R}}) \to \Gamma_c(\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}})'$ are even, linear, continuous, injective, and have dense image.*

[3] By assumption, X_0 is metrisable, and it is locally path-connected as a manifold. Hence, X_0 is second countable if and only if it has countably many connected components [29, Appendix].

Proof. For any compact $K \subseteq V_{\bar{0},\mathbb{R}}$, we have an injection $\Gamma_K(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}}) \to \mathscr{S}(V, V_{\bar{0},\mathbb{R}})$, which is continuous by definition of the topologies. Thus, so is $\Gamma_c(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}}) \to \mathscr{S}(V, V_{\bar{0},\mathbb{R}})$. This map has dense image, by [12, Chapter II, Section 2.5]. Hence, its transpose defines an injection $\mathscr{S}'(V, V_{\bar{0},\mathbb{R}}) \to \Gamma_c(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})'$ with dense image. □

Proof of Corollary C.8. By Proposition C.7, $\Gamma_c(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})$ is a nuclear LF space. Therefore, by [34, Proposition 50.6], $\Gamma_c(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})'$ is the locally convex projective limit of nuclear spaces, and thus itself nuclear, in view of Theorem 50.1 (*op. cit.*). As a subspace of a nuclear space, $\mathscr{S}'(V, V_{\bar{0},\mathbb{R}})$ is nuclear (*loc. cit.*). Hence, so is $\mathscr{S}(V, V_{\bar{0},\mathbb{R}})$, by Proposition 50.6 (*op. cit.*).

Any barrelled nuclear space is Montel, any nuclear space is reflexive, and the strong dual of a Montel space is Montel (hence, barrelled), so the claim follows. □

Using nuclearity, we derive along the lines of [34, proof of Theorem 51.6] the following corollary.

Corollary C.10. *Let X and Y be* cs *manifolds. There is a natural isomorphism of locally convex super-vector spaces $\Gamma(\mathcal{O}_X)\widehat{\otimes}_\pi\Gamma(\mathcal{O}_Y) \to \Gamma(\mathcal{O}_{X\times Y})$ where $\widehat{\otimes}_\pi$ denotes the completed projective tensor product topology.*

Let X be a *cs* manifold. The assignment $U \mapsto \Gamma_c(\mathcal{O}_X|_U)$ is a cosheaf [3], and its extension maps $\Gamma_c(\mathcal{O}_X|U) \to \Gamma_c(\mathcal{O}_X|_V)$ for open subsets $U \subseteq V \subseteq X_0$ are continuous, as follows from the definition of the topologies.

Thus, we have a presheaf $\mathcal{D}b_X$ on X_0, defined by

$$\mathcal{D}b_X(U) \coloneqq \Gamma_c(\mathcal{O}_X|_U)',$$

the topological dual space of $\Gamma_c(\mathcal{O}_X|_U)$. Because $\mathcal{O}_X$ is c-soft, it follows easily that $\mathcal{D}b_X$ is a sheaf. Sections of this sheaf are called *superdistributions* on X.

In view of Proposition C.7, when equipped with the strong topology, $\mathcal{D}b_X(U)$ is nuclear and Montel, and in particular, reflexive and barrelled.

C.2. **Vector-valued superfunctions.** In this subsection, we generalise the notion of a function with values in a locally convex space to the super case. Our motivation is Laurent Schwartz's approach to the study of the Laplace transform [31], which we will need to super-extend in order to prove the main result of this paper. However, the notion of vector-valued superfunctions is also useful in other contexts.

Rather than giving most general definition, which would appeal to some category of infinite-dimensional supermanifolds, we define vector-valued superfunctions *via* completed tensor products. This becomes tractable by a suitable extension of the formalism of S-valued points.

In what follows, let E denote a locally convex super-vector spaces and E' its strong continuous linear dual space. For any *cs* manifold S, we define

$$\mathcal{O}(S, E) \coloneqq \Gamma(\mathcal{O}_S)\widehat{\otimes}_\pi E,$$

where $\widehat{\otimes}_\pi$ denotes the completed projective tensor product, endowed with the standard grading. The elements of $\mathcal{O}(S, E)$ are called *E-valued superfunctions on S*. Observe that since $\Gamma(\mathcal{O}_S)$ is nuclear by Proposition C.7, we might have taken any other locally convex tensor product topology in the definition [34].

Proposition C.11. *Let E be a locally convex super-vector space. The assignment $S \mapsto \mathcal{O}(S, E)$ is a functor from* cs *manifolds to the category of locally convex super-vector spaces with even continuous linear maps.*

For any cs *manifolds S and T, there is a natural isomorphism*

$$\mathcal{O}(S \times T, E) = \mathcal{O}(S, \mathcal{O}(T, E))$$

of locally convex super-vector spaces.

Proof. The functoriality of $\mathcal{O}(-,E)$ follows from the definitions: Given a morphism $\phi : T \to S$ of *cs* manifolds, we may by the token of Proposition C.6 form

$$\mathcal{O}(\phi, E) \coloneqq \phi^\sharp \mathbin{\widehat{\otimes}_\pi} \mathrm{id}_E : \mathcal{O}(S,E) \to \mathcal{O}(T,E).$$

The second assertion is a consequence of Corollary C.10. $\square$

Definition C.12 (Values of vector valued superfunctions)**.** Let X be a *cs* manifold and $f \in \mathcal{O}(X,E)$. For any $x \in_S E$, we define

$$f(x) \coloneqq \mathcal{O}(x,E)(f) = \big(x^\sharp \mathbin{\widehat{\otimes}_\pi} \mathrm{id}_E\big)(f) \in \mathcal{O}(S,E),$$

and call this the *value of* f at the S-valued point x.

The following is immediate from the definitions.

Proposition C.13. *Let $X_1,\dots,X_n$ be* cs *manifolds, $E_1,\dots,E_n,F$ be locally convex spaces and $b : \prod_j E_j \to F$ an even continuous n-linear map. The assignment*

$$(f_1,e_1,\dots,f_n,e_n) : \prod\nolimits_j \Gamma(\mathcal{O}_{X_j}) \times E_j \mapsto f_1 \otimes \cdots \otimes f_b \otimes b(e_1,\dots,e_n)$$

extends uniquely to a continuous linear map

$$b : \mathcal{O}\big(\textstyle\prod_j X_j, \widehat{\bigotimes}_{\pi,j} E_j\big) \to \mathcal{O}\big(\textstyle\prod_j X_j, F\big)$$

which satisfies

$$b(f)(x_1,\dots,x_n) = b(f(x_1,\dots,x_n)) \in \mathcal{O}(S,F)$$

for any $f \in \mathcal{O}\big(\prod_j X_j, \widehat{\bigotimes}_{\pi,j} E_j\big)$ and $(x_1,\dots,x_n) \in_S \prod_j X_j$.

In particular, if $\langle\cdot,\cdot\rangle_E$ denotes the canonical pairing $E' \times E \to \mathbb{C}$, then we have

$$\langle \mu, f\rangle(x,y) = \langle \mu(x), f(y)\rangle \in \mathcal{O}(S,\mathbb{C}) = \Gamma(\mathcal{O}_S)$$

for any $\mu \in \mathcal{O}(X,E')$, $f \in \mathcal{O}(Y,E)$, and $(x,y) \in_S X \times Y$.

C.3. **Fourier transform on the Schwartz space $\mathscr{S}$.** In this subsection, we extend the classical theory of the Fourier transform on the Schwartz space to the super setting. Everything is more or less straightforward. However, we do not know a convenient reference, and consider it worthwhile to supply one.

In what follows, recall the facts and definitions from Appendix B.

Definition C.14. Let $(V, V_{\bar 0,\mathbb{R}})$ be a *cs* vector space of $\dim V = p|q$, endowed with a homogeneous basis (v_a, ν_b), where we assume $v_a \in V_{\bar 0,\mathbb{R}}$. Let (v^a, ν^b) be the dual basis. The Lebesgue density $|dv_0|$ is the unique translation invariant density on $V_{\bar 0,\mathbb{R}}$ such that the cube with side 1 spanned by (v^a) has volume 1.

Moreover, there is a unique Berezinian density $|Dv|$ on the cs supermanifold $L(V, V_{\bar 0,\mathbb{R}})$ associated with $(V, V_{\bar 0,\mathbb{R}})$, such that

$$\fint_{V_{\bar 0,\mathbb{R}}} \int_{L(V,V_{\bar 0,\mathbb{R}})} |Dv|\, f = |dv_0| \frac{\partial}{\partial \nu^q} \cdots \frac{\partial}{\partial \nu^1} f \quad \text{for all} \quad f \in \Gamma(\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}}).$$

We let $(V^*, V^*_{\bar 0,\mathbb{R}})$ be the dual *cs* vector space, with densities $|dv_0^*|$ and $|Dv^*|$ associated with the dual basis (v^a, ν^b).

The *standard retraction* $r = r_{V,V_{\bar 0,\mathbb{R}}}$ of $L(V, V_{\bar 0,\mathbb{R}})$ is defined by $r^\sharp(v^a) = v^a$. The definition is in fact independent of the choice of basis.

For $f \in \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$, we define the *Fourier transform* $\mathcal{F}(f) \in \mathscr{S}(V^*, V^*_{\bar 0,\mathbb{R}})$ by

$$\mathcal{F}(f) \coloneqq \frac{1}{(2\pi)^{p/2}} \int_{L(V,V_{\bar 0,\mathbb{R}})} |Dv|\, e^{-i\langle\cdot,v\rangle} f(v),$$

where $\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{C}$ denotes the canonical pairing. For $f \in \mathscr{S}(V_{\bar 0,\mathbb{R}})$, we normalise the ordinary Fourier transform $\mathcal{F}_0(f) \in \mathscr{S}(V^*_{\bar 0,\mathbb{R}})$ by

$$\mathcal{F}_0(f) := \frac{1}{(2\pi)^{p/2}} \int_{V_{\bar 0,\mathbb{R}}} |dv_0|\, e^{-i\langle\cdot,v_0\rangle} f(v_0).$$

For $f \in \mathscr{S}(V^*, V^*_{\bar 0,\mathbb{R}})$, we define the *Fourier cotransform* $\check{\mathcal{F}}(f) \in \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ by

$$\check{\mathcal{F}}(f) := \frac{(-i)^q(-1)^{\frac{1}{2}q(q+1)}}{(2\pi)^{p/2}} \int_{L(V^*,V^*_{\bar 0,\mathbb{R}})} |Dv^*|\, e^{i\langle v^*,\cdot\rangle} f(v^*).$$

We normalise the ordinary Fourier cotransform $\check{\mathcal{F}}_0(f) \in \mathscr{S}(V_{\bar 0,\mathbb{R}})$ of $f \in \mathscr{S}(V^*_{\bar 0,\mathbb{R}})$ by

$$\check{\mathcal{F}}_0(f) = \frac{1}{(2\pi)^{p/2}} \int_{V^*_{\bar 0,\mathbb{R}}} |dv_0^*|\, e^{i\langle v_0^*,\cdot\rangle} f(v_0^*),$$

so that $\check{\mathcal{F}}_0 = \mathcal{F}_0^{-1}$ by the classical Fourier inversion theorem [5].

From the classical theory, one deduces easily that $\mathcal{F}$ and $\check{\mathcal{F}}$ are continuous.

Lemma C.15. *Let $(U, U_{\bar 0,\mathbb{R}})$ be a* cs *vector space and $\langle\cdot,\cdot\rangle$ denote the canonical pairing $U^* \times U \to \mathbb{C}$. If (u_a) is a homogeneous basis of U with dual basis (u^a), so that $u^b(u_a) = \delta_{ab}$, then as a superfunction on $L(U^*, U^*_{\bar 0,\mathbb{R}}) \times L(U, U_{\bar 0,\mathbb{R}})$, we have*

$$\langle\cdot,\cdot\rangle = \sum_a u_a \otimes u^a.$$

Proof. Let S be any *cs* manifold, and let $u^* \in_S L(U^*, U^*_{\bar 0,\mathbb{R}})$, $u \in_S L(U, U_{\bar 0,\mathbb{R}})$, where

$$u^* = \sum_a f_a u^a \quad \text{and} \quad u = \sum_a g^a u_a$$

under the identification $U(S) = (\Gamma(\mathcal{O}_S) \otimes U)_{\bar 0,\mathbb{R}}$ (and similarly for U^*). Then

$$\langle u^*, u\rangle = \sum_{ab} (-1)^{|u^a||u_b|} f_a g^b u^a(u_b) = \sum_a (-1)^{|u_a|} f_a g^a = \operatorname{str}(f_a g^b).$$

On the other hand,

$$\Bigl(\sum\nolimits_a u_a \otimes u^a\Bigr)(u^*, u) = \sum_{abc} (-1)^{|u_a||u^b| + |u^a||u_c|} f_b g^c u_a(u^b) u^a(u_c) = \operatorname{str}(f_a g^b),$$

since $u_a(u^b) = (-1)^{|u_a|}\delta_{ab}$. ☐

Lemma C.16. *For any $f \in \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$, $g \in \mathscr{S}(V^*, V^*_{\bar 0,\mathbb{R}})$, and $a = 1, \dots, q$, we have*

$$\mathcal{F}(\nu^a f) = (-1)^q i \frac{\partial}{\partial \nu_a} \mathcal{F}(f), \qquad \check{\mathcal{F}}(\nu_a g) = (-1)^q i \frac{\partial}{\partial \nu^a} \check{\mathcal{F}}(g),$$

$$\mathcal{F}\Bigl(\frac{\partial}{\partial \nu^a} f\Bigr) = -(-1)^q i \nu_a \mathcal{F}(f), \qquad \check{\mathcal{F}}\Bigl(\frac{\partial}{\partial \nu_a} g\Bigr) = -(-1)^q i \nu^a \check{\mathcal{F}}(g).$$

Proof. By Lemma C.15, we have

$$\frac{\partial}{\partial \nu_a}\langle\cdot,\cdot\rangle = \nu^a \quad \text{and} \quad \frac{\partial}{\partial \nu^a}\langle\cdot,\cdot\rangle = -\nu_a,$$

so that

$$\frac{\partial}{\partial \nu_a} e^{\pm i\langle\cdot,\cdot\rangle} = \sum_{n=0}^{\infty} \frac{(\pm i)^{n+1}}{n!} \nu^a \langle\cdot,\cdot\rangle^n = \pm i \nu^a e^{\pm i\langle\cdot,\cdot\rangle}$$

and $\frac{\partial}{\partial\nu^a}e^{\pm i\langle\cdot,\cdot\rangle} = \mp i\nu_a e^{\pm i\langle\cdot,\cdot\rangle}$. Then

$$\begin{aligned}\mathcal{F}(\nu^a f) &= \int_{L(V,V_{\bar 0,\mathbb{R}})} |Dv|\, \nu^a e^{-i\langle\cdot,v\rangle} f(v)\\ &= i\int_{L(V,V_{\bar 0,\mathbb{R}})} |Dv|\, \frac{\partial}{\partial\nu_a} e^{-i\langle\cdot,v\rangle} f(v) = (-1)^q i\frac{\partial}{\partial\nu_a}\mathcal{F}(f),\end{aligned}$$

and similarly $\check{\mathcal{F}}(\nu_a g) = (-1)^q i\frac{\partial}{\partial\nu^a}\check{\mathcal{F}}(g)$. Using integration by parts, we see

$$\mathcal{F}\Big(\frac{\partial}{\partial\nu^a}f\Big) = -\int_{L(V,V_{\bar 0,\mathbb{R}})} |Dv|\, \frac{\partial}{\partial\nu^a} e^{-i\langle\cdot,v\rangle} f(v) = -(-1)^q i\nu_a\mathcal{F}(f),$$

and similarly $\check{\mathcal{F}}\big(\frac{\partial}{\partial\nu_a}g\big) = -(-1)^q i\nu^a\check{\mathcal{F}}(g)$. □

Proposition C.17. *The linear map $\mathcal{F} : \mathscr{S}(V, V_{\bar 0,\mathbb{R}}) \to \mathscr{S}(V^*, V^*_{\bar 0,\mathbb{R}})$ is an isomorphism of locally convex vector spaces of parity $\equiv q\,(2)$, with inverse $\check{\mathcal{F}}$.*

Note that when considered on the level of Berezinian densities instead of functions, the Fourier transform is an even map.

Proof of Proposition C.17. The idea is to reduce to the classical case by taking derivatives, as is done for $\dim V = 0|q$ in Ref. [13, Chapter 7].

Let $f \in \mathscr{S}(V_{\bar 0,\mathbb{R}})$, considered as an element of $\mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ *via* the standard retraction of V. Letting v^* denote the generic point of $L(V^*, V^*_{\bar 0,\mathbb{R}})$, the proof of Lemma C.16 shows that

$$\begin{aligned}\mathcal{F}(f)(v^*) &= \frac{1}{(2\pi)^{p/2}}\int_{V_{\bar 0,\mathbb{R}}} |dv_0|\, j^\sharp_{L(V,V_{\bar 0,\mathbb{R}})_0}\Big(\frac{\partial}{\partial\nu^q}\cdots\frac{\partial}{\partial\nu^1}e^{-i\langle v^*,\cdot\rangle}\Big)(v_0)f(v_0)\\ &= i^q(-1)^{\frac12 q(q-1)}\nu_1\cdots\nu_q\,\mathcal{F}_0(f),\end{aligned}$$

where $\mathcal{F}_0(f)$ is considered as a superfunction on $L(V^*, V^*_{\bar 0,\mathbb{R}})$ *via* the standard retraction. It follows that

$$\check{\mathcal{F}}\mathcal{F}(f) = \frac{1}{(2\pi)^{p/2}}\int_{V^*_{\bar 0,\mathbb{R}}} |dv_0^*|\, e^{i\langle v_0^*,v_0\rangle} j^\sharp_{L(V^*,V^*_{\bar 0,\mathbb{R}})_0}\Big(\frac{\partial}{\partial\nu_q}\cdots\frac{\partial}{\partial\nu_1}\nu_1\cdots\nu_q\,\mathcal{F}_0(f)\Big)(v_0^*) = f,$$

since $e^{i\langle\cdot,\cdot\rangle}$ is even, and by the classical Fourier inversion formula [5].

If now $f \in \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ is arbitrary, then by Lemma C.16,

$$\check{\mathcal{F}}\mathcal{F}(\nu^a f) = (-1)^q i\,\check{\mathcal{F}}\Big(\frac{\partial}{\partial\nu_a}\mathcal{F}(f)\Big) = \nu^a\check{\mathcal{F}}\mathcal{F}(f).$$

This reduces the proof of the equation $\check{\mathcal{F}}\mathcal{F} = \mathrm{id}$ to the subspace $\mathscr{S}(V_{\bar 0,\mathbb{R}})$, which was treated above. For the converse composition, one proceeds analogously. □

Definition C.18. For $f, g \in \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$, define the *convolution* $f * g \in \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$ by demanding that

$$\int_V |Dv|\,(f*g)(v)h(v) = \int_{V\times V} |Dv_1||Dv_2|\, f(v_1)g(v_2)h(v_1+v_2)$$

for any tempered superfunction h. Here, we use the short-hand $\int_V$ for $\int_{L(V,V_{\bar 0,\mathbb{R}})}$. Then, for any *cs* manifold S, and $x \in_S L(V, V_{\bar 0,\mathbb{R}})$,

$$(f*g)(x) = \int_V |Dv|\, f(v)g(x-v).$$

The following lemma, which is an easy consequence of the Leibniz rule and Hölder's inequality, shows that indeed $f * g \in \mathscr{S}(V, V_{\bar 0,\mathbb{R}})$.

Lemma C.19. *For all $I\subseteq\{1,\dots,q\}$, there exist $D_{I1},D_{I2}\in S(V^*)$, such that*

$$\left|\int_{L(V,V_{\bar 0,\mathbb{R}})}|Dv|\,(fg)(v)\right|\leqslant\sum_I\sup_{v_0\in V_{\bar 0,\mathbb{R}}}\bigl|f(D_{I1};v_0)\bigr|\int_{V_{\bar 0,\mathbb{R}}}|dv_0|\,\bigl|g(D_{I2};v_0)\bigr|,$$

for any superfunctions f,g such that all integrals in question converge absolutely.

The behaviour of convolution products under Fourier transform carries over to the super case. The non-trivial signs are again an artifact introduced by considering the Fourier transform on the level of functions rather than of Berezinian densities.

Lemma C.20. *For any $f,g\in\mathscr{S}(V,V_{\bar 0,\mathbb{R}})$, we have*

$$\mathcal{F}(f*g)=(-1)^{q|f|}(2\pi)^{p/2}\mathcal{F}(f)\mathcal{F}(g).$$

Proof. By the definition of $f*g$, and writing $\int_V$ for $\int_{L(V,V_{\bar 0,\mathbb{R}})}$, we compute

$$\begin{aligned}\mathcal{F}(f*g)&=\frac{1}{(2\pi)^{p/2}}\int_{V\times V}|Dv_1||Dv_2|\,e^{-i\langle\cdot,v_1+v_2\rangle}f(v_1)g(v_2)\\&=\frac{(-1)^{q|f|}}{(2\pi)^{p/2}}\int_V|Dv|\,e^{-i\langle\cdot,v\rangle}f(v)\cdot\int_V|Dv|\,e^{-i\langle\cdot,v\rangle}g(v)\\&=(-1)^{q|f|}(2\pi)^{p/2}\mathcal{F}(f)\mathcal{F}(g).\end{aligned}$$

Here, the equality

$$e^{-i\langle u,v_1+v_2\rangle}=e^{-i\langle u,v_1\rangle}e^{-i\langle u,v_2\rangle}$$

for all $u,v_j\in_S L(V,V_{\bar 0,\mathbb{R}})$ follows as usual using Cauchy summation, since the exponential series in question converge absolutely in $\Gamma(\mathcal{O}_S)$, in view of Corollary C.5 and the (elementary) fact that complete locally m-convex algebras admit a functional calculus for entire functions, *cf.* Ref. [25]. This proves the claim. □

Proposition C.21. *Let $f,g\in\mathscr{S}(V,V_{\bar 0,\mathbb{R}})$. Then*

$$\int_{V^*}|Dv^*|\,\mathcal{F}(f)(v^*)\mathcal{F}(g)(v^*)=(-1)^{q|f|}i^q(-1)^{\frac12 q(q-1)}\int_V|Dv|\,f(v)g(-v),$$

where we write $\int_V$ for $\int_{L(V,V_{\bar 0,\mathbb{R}})}$.

Proof. We have

$$\check{\mathcal{F}}(h)(0)=\frac{(-i)^q(-1)^{\frac12 q(q-1)}}{(2\pi)^{p/2}}\int_{V^*}|Dv^*|\,h(v^*)$$

for all $h\in\mathscr{S}(V^*,V^*_{\bar 0,\mathbb{R}})$, so by Lemma C.20 and Proposition C.17,

$$\begin{aligned}\int_{V^*}|Dv^*|\,\mathcal{F}(f)(v^*)\mathcal{F}(g)(v^*)&=\frac{(-1)^{q|f|}}{(2\pi)^{p/2}}\int_{V^*}|Dv^*|\,\mathcal{F}(f*g)(v^*)\\&=(-1)^{q|f|}i^q(-1)^{\frac12 q(q-1)}\check{\mathcal{F}}\mathcal{F}(f*g)(0)\\&=(-1)^{q|f|}i^q(-1)^{\frac12 q(q-1)}\int_V|Dv|\,f(v)g(-v),\end{aligned}$$

which was our assertion. □

Finally, we define the Fourier (co)transform on $\mathscr{S}'(V,V_{\bar 0,\mathbb{R}})$ by duality.

Definition C.22. For any homogeneous $\mu\in\mathscr{S}'(V,V_{\bar 0,\mathbb{R}})$, we define the distributional *Fourier transform* $\mathcal{F}(\mu)\in\mathscr{S}'(V^*,V^*_{\bar 0,\mathbb{R}})$ by

$$\langle\mathcal{F}(\mu),\mathcal{F}(f)\rangle\coloneqq(-1)^{q|\mu|}i^q(-1)^{\frac12 q(q-1)}\langle\mu,\check f\rangle\quad\text{for all}\quad f\in\mathscr{S}(V,V_{\bar 0,\mathbb{R}})$$

where $\check{f}(v) = f(-v)$ for all $v \in_S L(V, V_{\bar{0},\mathbb{R}})$ and any *cs* manifold S. Similarly, define, for homogeneous $\nu \in \mathscr{S}'(V^*, V^*_{\bar{0},\mathbb{R}})$, the *Fourier cotransform* $\check{\mathcal{F}}(\nu) \in \mathscr{S}'(V, V_{\bar{0},\mathbb{R}})$ by

$$\langle \check{\mathcal{F}}(\mu), \check{\mathcal{F}}(g) \rangle := (-1)^{q|\nu|}(-i)^q(-1)^{\frac{1}{2}q(q-1)} \langle \nu, \check{g} \rangle \quad \text{for all } \; g \in \mathscr{S}(V^*, V^*_{\bar{0},\mathbb{R}}).$$

The following is immediate.

Corollary C.23. *The Fourier transform* $\mathcal{F} : \mathscr{S}'(V, V_{\bar{0},\mathbb{R}}) \to \mathscr{S}'(V^*, V^*_{\bar{0},\mathbb{R}})$ *is an isomorphism of locally convex vector spaces, of parity* $\equiv q\,(2)$*, with inverse* $\check{\mathcal{F}}$.

Again, the parity problems disappear if instead we consider the Fourier transform on the level of tempered generalised functions (*i.e.* the dual of the space of Berezinian densities of Schwartz class).

C.4. **The Paley–Wiener space $\mathscr{Z}$.** We will also need to consider the Fourier transform in situations which do not exhibit the same self-duality as the case of the Schwartz space. A useful receptacle will be the so-called Paley–Wiener space, which arises by Fourier transform of compactly supported smooth functions.

Definition C.24. Fix a positive inner product $(\cdot|\cdot)_V$ on $V_{\bar{0},\mathbb{R}}$. Write $\|\cdot\|_V$ for the associated norm function.

Denote by $L(V)$ the complex supermanifold associated with the complex super-vector space V, with sheaf of superfunctions $\mathcal{O}_V$. We denote by $\mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$ the following subspace of $\Gamma(\mathcal{O}_V)$,

$$\mathscr{Z}(V, V_{\bar{0},\mathbb{R}}) := \big\{ f \in \Gamma(\mathcal{O}_V) \;\big|\; \exists R > 0\, \forall D \in S(V)\,,\, p \in S(V^*) \colon z_{R,D,p}(f) < \infty \big\},$$

where

$$z_{R,D,p}(f) := \sup\nolimits_{v \in V_{\bar{0}}} \big| e^{-R\|\Im v\|_V} (p \cdot f)(D; v) \big|$$

for all $R > 0$, $p \in S(V^*)$, $D \in S(V)$ and $f \in \Gamma(\mathcal{O}_V)$. Here, $\Im v := \frac{1}{2i}(v - \bar{v})$ where $\bar{v}$ is the complex conjugate of v with respect to the real form $V_{\bar{0},\mathbb{R}}$ of $V_{\bar{0}}$.

Thus, $\mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$ consists of holomorphic superfunctions of exponential type; it is called the *Paley–Wiener space* of $(V, V_{\bar{0},\mathbb{R}})$. We also consider for fixed $R > 0$ the subspace $\mathscr{Z}_R(V, V_{\bar{0},\mathbb{R}}) \subseteq \Gamma(\mathcal{O}_V)$ defined by the requirement that all $z_{R,D,p}$, for p and D arbitrary, are finite. With the topology induced by these seminorms, $\mathscr{Z}_R(V, V_{\bar{0},\mathbb{R}})$ is a Fréchet space. We endow $\mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$ with the locally convex inductive limit topology of the spaces $\mathscr{Z}_R(V, V_{\bar{0},\mathbb{R}})$. Obviously, the restriction map $\Gamma(\mathcal{O}_V) \to \Gamma(\mathcal{O}_{V, V_{\bar{0},\mathbb{R}}})$ induces a continuous injection $\mathscr{Z}(V, V_{\bar{0},\mathbb{R}}) \to \mathscr{S}(V, V_{\bar{0},\mathbb{R}})$.

Proposition C.25 (Paley–Wiener)**.** *Let* $\check{\mathcal{F}} : \mathscr{S}(V, V_{\bar{0},\mathbb{R}}) \to \mathscr{S}(V^*, V^*_{\bar{0},\mathbb{R}})$ *be the Fourier cotransform of* $(V^*, V^*_{\bar{0},\mathbb{R}})$ *and* $f \in \mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$*. Then* $\check{\mathcal{F}}(f) \in \Gamma_c(\mathcal{O}_{V^*, V^*_{\bar{0},\mathbb{R}}})$*, and* $f \in \mathscr{Z}_R(V, V_{\bar{0},\mathbb{R}})$ *if and only if* $\operatorname{supp} \check{\mathcal{F}}(f) \subseteq B_{V^*}(0, R)$*, where*

$$B_{V^*}(0, R) = \big\{ v^* \in V^*_{\bar{0},\mathbb{R}} \;\big|\; \|v^*\|_{V^*} \leqslant R \big\},$$

and $\|\cdot\|_{V^*}$ *denotes the norm dual to* $\|\cdot\|_V$*. Moreover, this sets up an isomorphism of locally convex spaces* $\Gamma_{B_{V^*}(0,R)}(\mathcal{O}_{V^*}) \cong \mathscr{Z}_R(V, V_{\bar{0},\mathbb{R}})$.

Proof. In view of Lemma C.16, the proof is reduced to the case of $f \in \mathscr{Z}(V_{\bar{0},\mathbb{R}})$. Then by the proof of Proposition C.17, $\check{\mathcal{F}}(f) = \nu_1 \cdots \nu_q \check{\mathcal{F}}_0(f)$ which has support in $B_{V^*}(0, R)$ if and only if this is the case for $\check{\mathcal{F}}_0(f)$. Hence, the statement reduces to the classical Paley–Wiener theorem, *v.* Refs. [5, 6, 11, 32]. □

Corollary C.26. *The locally convex spaces* $\mathscr{Z}_R(V, V_{\bar{0},\mathbb{R}})$ *are nuclear Fréchet, and* $\mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$ *is nuclear LF. In particular, both spaces are complete, reflexive, barrelled, Montel, and bornological.*

Proof. The topology on $\mathscr{Z}_R(V, V_{\bar{0},\mathbb{R}})$ (resp. $\mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$) is the one induced *via* the Fourier transform from $\Gamma_{B_{V^*}(0,R)}(\mathcal{O}_{V^*})$ (resp. $\Gamma_c(\mathcal{O}_{V^*})$), and the latter is nuclear Fréchet (resp. nuclear LF) and Montel by Proposition C.7. □

Definition C.27. For any homogeneous $\mu \in \mathscr{Z}'(V)$, we define the distributional *Fourier transform* $\mathcal{F}(\mu) \in \Gamma(\mathcal{D}b_{V^*,V^*_{\bar{0},\mathbb{R}}})$ by

$$\langle \mathcal{F}(\mu), \mathcal{F}(f)\rangle := (-1)^{q|\mu|} i^q (-1)^{\frac{1}{2}q(q-1)} \langle \mu, \check{f}\rangle \quad \text{for all} \quad f \in \mathscr{Z}(V, V_{\bar{0},\mathbb{R}}).$$

Here, we write $\mathcal{D}b_{V^*,V^*_{\bar{0},\mathbb{R}}} := \mathcal{D}b_{L(V^*,V^*_{\bar{0},\mathbb{R}})}$. Similarly, we define, for $\nu \in \Gamma(\mathcal{D}b_{V^*,V^*_{\bar{0},\mathbb{R}}})$, the *Fourier cotransform* $\check{\mathcal{F}}(\nu) \in \mathscr{Z}'(V)$ by

$$\langle \check{\mathcal{F}}(\mu), \check{\mathcal{F}}(g)\rangle := (-1)^{q|\nu|} (-i)^q (-1)^{\frac{1}{2}q(q-1)} \langle \nu, \check{g}\rangle \quad \text{for all} \quad g \in \Gamma_c(\mathcal{O}_{V^*,V^*_{\bar{0},\mathbb{R}}}).$$

Corollary C.28. *The Fourier transform* $\mathcal{F} : \mathscr{Z}'(V) \to \Gamma(\mathcal{D}b_{V^*,V^*_{\bar{0},\mathbb{R}}})$ *is an isomorphism of locally convex vector spaces, of parity* $\equiv q\,(2)$*, with inverse* $\check{\mathcal{F}}$*.*

C.5. **Laplace transforms.** In this subsection, we give an account of the basics of the Laplace transform. The two main results are that the Laplace transform is injective, and that under mild conditions it can be computed as an integral. Another point, which we discuss at some length, is the extension of generalised superfunctions as functionals to certain larger spaces of test superfunctions, depending on the domains of definition of their Laplace transforms.

Essentially, we follow the classical exposition by Schwartz [31], although we also need to consider the Laplace transform for functionals on $\mathscr{Z}$ (as in Ref. [16]). Moreover, a rigorous account of this theory for superspaces needs to use S-valued points; in this, we follow the exposition given in Appendix C.2.

Definition C.29. A locally convex super-vector space E is called a *test space* for $(V, V_{\bar{0},\mathbb{R}})$ if it is one of $\Gamma_c(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})$, $\mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$, or $\mathscr{S}(V, V_{\bar{0},\mathbb{R}})$. In this case, $\check{E} := \mathcal{F}(E)$ (where $\mathcal{F}$ is the Fourier transform on $(V, V_{\bar{0},\mathbb{R}})$) is called the *the dual test space* (for $(V^*, V^*_{\bar{0},\mathbb{R}})$); one also has $\check{E} = \check{\mathcal{F}}(E)$, where $\check{\mathcal{F}}$ is the Fourier cotransform on $(V^*, V^*_{\bar{0},\mathbb{R}})$. Notice that all test spaces are contained as dense subspaces in $\mathscr{S}(V, V_{\bar{0},\mathbb{R}})$

The strong dual E' of a test space is called a space of *generalised functions* for $(V, V_{\bar{0},\mathbb{R}})$. Notice that $\mathscr{S}'(V, V_{\bar{0},\mathbb{R}})$ is contained as a dense subspace in any space of generalised functions E'. The set of $\mathcal{M}(E)$ of all functions $f \in \Gamma(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})$ such that $f \cdot E \subseteq E$ in $\Gamma(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})$ is called the *multiplier space* of E.

For $E = \Gamma_c(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})$, we have $\mathcal{M}(E) = \Gamma(\mathcal{O}_{V,V_{\bar{0},\mathbb{R}}})$. For $E = \mathscr{S}(V, V_{\bar{0},\mathbb{R}})$, we have $\mathcal{M}(E) = \mathcal{T}(V, V_{\bar{0},\mathbb{R}})$, the space of tempered superfunctions [5]. Finally, for $E = \mathscr{Z}(V, V_{\bar{0},\mathbb{R}})$, we have [11]

$$\mathcal{M}(E) = \big\{ f \in \Gamma(\mathcal{O}_V) \;\big|\; \forall D \in S(V) \exists R, N > 0 : z_{R,D,-N}(f) < \infty \big\},$$

where

$$z_{R,D,-N}(f) := \sup_{v \in V_{\bar{0}}} \|v\|_V^{-N} e^{-R\|\Im v\|_V} |f(D; v)|.$$

Notice that $\mathcal{M}(E)$ contains the space $S(V^*)$ of superpolynomials for any E.

Let $E \subseteq F$ be test spaces and $\mu \in E'$. If $z \in_S L(V^*)_{cs}$, then we write

$$e^{-\langle z, \cdot\rangle} \mu \in_S F'$$

if the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle x, v\rangle^k \mu \tag{C.1}$$

converges in $\mathcal{O}(S,E')$, and its limit, denoted by $e^{-\langle z,\cdot\rangle}\mu$, lies in the subspace $\mathcal{O}(S,F')$. Here, v denotes the generic point of $L(V,V_{\bar 0,\mathbb{R}})$. We define

$$\gamma_F(\mu)_S := \big\{x \in_S L(V^*,V_{\bar 0,\mathbb{R}}^*) \;\big|\; e^{-\langle x,\cdot\rangle}\mu \in_S F'\big\}.$$

We also write $\gamma_{\mathscr{D}}$, $\gamma_{\mathscr{S}}$, and $\gamma_{\mathscr{Z}}$, for the case of $E = \Gamma_c(\mathcal{O}_{V,V_{\bar 0,\mathbb{R}}})$, $E = \mathscr{S}(V,V_{\bar 0,\mathbb{R}})$, and $E = \mathscr{Z}(V,V_{\bar 0,\mathbb{R}})$, respectively.

We remark that for any $z = x+iy \in_S L(V^*)_{cs}$, $x,y \in_S L(V^*,V_{\bar 0,\mathbb{R}}^*)$, we have

$$e^{-\langle z,\cdot\rangle}\mu \in_S F' \quad\Leftrightarrow\quad x \in \gamma_F(\mu)_S. \tag{C.2}$$

Definition C.30. Let $E \subseteq F$ be test spaces for $(V,V_{\bar 0,\mathbb{R}})$ and $\mu \in E'$. For any $x \in \gamma_F(\mu)_S$, we define the *Laplace transform*

$$\mathscr{L}(\mu)(x) := \mathcal{F}(e^{-\langle x,\cdot\rangle}\mu) \in \mathcal{O}(S,\check{F}').$$

Lemma C.31. *Let $E \subseteq F$ be test spaces for $(V,V_{\bar 0,\mathbb{R}})$ and $\mu \in E'$. If $x \in \gamma_F(\mu)_S$ and $t : T \to S$, then $x(t) \in \gamma_F(\mu)_T$ and*

$$\mathscr{L}(\mu)(x(t)) = (\mathscr{L}(\mu)(x))(t).$$

Proof. Applying Proposition C.6 to exchange $t^\sharp \mathbin{\widehat{\otimes}}_\pi \mathrm{id}$ with the limit of the series, we find $x(t) \in \gamma_F(\mu)_T$. The second assertion follows from Proposition C.13 and the continuity of the Fourier transform. □

We now relate the thus defined Laplace transform of generalised superfunctions to the Laplace transform of ordinary generalised functions. First, observe that the following is immediate by the Leibniz rule.

Lemma C.32. *Let $E \subseteq F$ be test spaces and $\mu \in E'$. Then*

$$\gamma_F(\nu^b\mu)_S \supseteq \gamma_F(\mu)_S \quad\textit{and}\quad \gamma_F\big(\tfrac{\partial}{\partial\nu_b}\mu\big)_S \supseteq \gamma_F(\mu)_S.$$

Denote the standard retraction of $L(V,V_{\bar 0,\mathbb{R}})$ by $r_{V,V_{\bar 0,\mathbb{R}}} : L(V,V_{\bar 0,\mathbb{R}}) \to V_{\bar 0,\mathbb{R}}$. Let E be a test space for $(V,V_{\bar 0,\mathbb{R}})$ and E_0 the corresponding test space for $V_{\bar 0,\mathbb{R}}$; that is, $E_0 = \mathscr{Z}(V_{\bar 0,\mathbb{R}})$ if $E = \mathscr{Z}(V,V_{\bar 0,\mathbb{R}})$, *etc.* Then $r^\sharp_{V,V_{\bar 0,\mathbb{R}}}(E_0) \subseteq E$, so

$$\langle r_{V,\sharp}(\mu), f\rangle := \langle \mu, r^\sharp_{V,V_{\bar 0,\mathbb{R}}}(f)\rangle \quad\text{for all}\quad \mu \in E'\,,\ f \in E_0.$$

defines a continuous even linear map $r_{V,V_{\bar 0,\mathbb{R}},\sharp} : E' \to E_0'$.

Proposition C.33. *Let $E \subseteq F$ be test spaces and $\mu \in E'$. Then*

$$\gamma_F(\mu)_S = \bigcap_{k=0}^{q} \bigcap_{1\leqslant b_1<\cdots<b_k\leqslant q} \gamma_{F_0}\big(r_{V,V_{\bar 0,\mathbb{R}},\sharp}(\nu^{b_1}\cdots\nu^{b_k}\mu)\big)_S.$$

Proof. The statement is immediate from the Taylor expansion

$$\langle \mu, f\rangle = \sum_{k=0}^{q} \sum_{B=(1\leqslant b_1<\cdots<b_k\leqslant q)} \pm\Big\langle r_{V,V_{\bar 0,\mathbb{R}},\sharp}(\nu^B\mu), j^\sharp_{L(V,V_{\bar 0,\mathbb{R}})}\Big(\frac{\partial^{b_1+\cdots+b_k}}{\partial\nu_B} f\Big)\Big\rangle, \tag{C.3}$$

which can be applied to $\langle e^{-\langle x,\cdot\rangle}\cdot\mu, f\rangle$. □

If X is any *cs* manifold and $j : Y \to X$ is an embedding, then if a morphism $S \to X$ factors through j, then it does so uniquely. Hence, for any open subspace U of X, $U(S)$ may be considered a as subset of $X(S)$ for any S, and the totality of these subsets form a topology on $X(S)$. For this topology, any morphism $f : X \to Y$ induces continuous maps $X(S) \to Y(S)$ on S-valued points; moreover, if $j : Y \to X$ is an embedding, then j is open if and only if $Y(S) \subseteq X(S)$ is open for every S.

In what follows, given an open subspace $\gamma \subseteq L(V^*,V_{\bar 0,\mathbb{R}}^*)$, we denote by $T(\gamma)$ the open subspace $\gamma + L(V^*,V_{\bar 0,\mathbb{R}}^*)$ of $L(V^*)$.

Theorem C.34. *Let $E \subseteq F$ be test spaces for $(V,V_{\bar 0,\mathbb{R}})$ and $\mu \in E'$.*

(i). *There exists a unique open subspace $\gamma_F^\circ(\mu)$ of $L(V^*, V_{\bar 0,\mathbb{R}}^*)$ such that for any* cs *manifold S, $\gamma_F^\circ(\mu)(S)$ is the interior of $\gamma_F(\mu)_S$ in $L(V^*, V_{\bar 0,\mathbb{R}}^*)(S)$.*

(ii). *Assume that $\gamma_{\mathscr{S}}^\circ(\mu) \neq \varnothing$ and let $T(\mu) := T(\gamma_{\mathscr{S}}^\circ(\mu))$. Then there is a unique $f \in \Gamma(\mathcal{O}_{T_{\mathscr{S}}(\mu)})$ such that*

$$\int_{L(V^*,V_{\bar 0,\mathbb{R}}^*)} |Dy|\, f(x+iy)\varphi(y) = \langle \mathscr{L}(\mu)(x), \varphi\rangle \tag{C.4}$$

for any $\varphi \in \mathscr{S}(V^*, V_{\bar 0,\mathbb{R}}^*)$ and $x \in_S \gamma_{\mathscr{S}}^\circ(\mu)$, in the sense that the integral on the left hand side converges absolutely and the equality holds. Moreover, for any $D \in S(V^*)$, the function $f(D; z)$ is bounded on $T(K)$ by some polynomial in $\Im z$, for every compact subset $K \subseteq \gamma_{\mathscr{S}}^\circ(\mu)_0$.

(iii). *Given an open subspace $\gamma \subseteq L(V^*, V_{\bar 0,\mathbb{R}}^*)$ and $f \in \Gamma(\mathcal{O}_{T(\gamma)})$, a tempered superdistribution $\mu \in \mathscr{S}'(V, V_{\bar 0,\mathbb{R}})$ exists such that $\gamma \subseteq \gamma_{\mathscr{S}}^\circ(\mu)$ and Equation C.4 holds, if and only if for any $D \in S(V^*)$, the function $f(D; z)$ is bounded on $T(K)$ by some polynomial in $\Im z$, for every compact subset $K \subseteq \gamma_0$. In this case, μ is unique.*

Proof. (i). Unicity is obvious, and so we check existence. In view of Proposition C.33, we may assume that $V = V_{\bar 0}$ and $\mu \in E' = E_0'$ is an ordinary generalised function. In this case, we let $\gamma_F^\circ(\mu)$ be the set of all $x \in V_{\bar 0,\mathbb{R}}$ such that $e^{-\langle y,\cdot\rangle}\cdot\mu \in F'$ for all y a some neighbourhood of x. In general, $\gamma_F^\circ(\mu)$ will be defined by

$$\gamma_F^\circ(\mu) := \bigcap_{k=0}^{q} \bigcap_{1\leqslant b_1<\cdots<b_k\leqslant q} \gamma_{F_0}^\circ\big(r_{V,V_{\bar 0,\mathbb{R}},\sharp}(\nu^{b_1}\cdots\nu^{b_k}\mu)\big).$$

If $\gamma_F(\mu)_S^\circ \neq \varnothing$ where $S_0 \neq \varnothing$, then there exists an open subset $U \subseteq V_{\bar 0,\mathbb{R}}$ such that for any $y \in_S U$, we have $e^{-\langle y,\cdot\rangle}\cdot\mu \in \mathcal{O}(S, F')$. In particular, $e^{-\langle u,\cdot\rangle}\cdot\mu \in F'$ for any $u \in U$ which appears as the value of some $y \in_S U$. But since the image of S_0 is non-empty, any $u \in U$ appears in this way (by considering constant morphisms).

Then by Equations (C.2) and (C.1), the map $z \mapsto e^{-\langle z,\cdot\rangle}\cdot\mu : U + iV_{\bar 0,\mathbb{R}} \to F'$ is (strongly) holomorphic, and in particular, if u denotes the generic point of U, we have $e^{-\langle z,\cdot\rangle}\cdot\mu \in \mathcal{O}(U, F') = \mathcal{C}^\infty(U)\widehat{\otimes}_\pi F'$. In other words, $U \subseteq \gamma_F^\circ(\mu)$.

Thus, any S-valued point of $L(V, V_{\bar 0,\mathbb{R}})$ in the interior of $\gamma(\mu)_S$ is the specialisation of the identity of some open subset of $\gamma_F^\circ(\mu)$, and this proves the claim.

(ii)-(iii). The statements reduce by the token of Equation (C.3) to the classical case, which is given in [31, Proposition 6]. $\square$

The following is immediate by combining items (ii) and (iii) of Theorem C.34.

Corollary C.35. *Let E be a test space for $(V, V_{\bar 0,\mathbb{R}})$ and $\mu \in E'$. If $\gamma_{\mathscr{S}}^\circ(\mu) \neq \varnothing$, then $\mu \in \mathscr{S}'(V, V_{\bar 0,\mathbb{R}})$, in the sense that the functional extends continuously to $\mathscr{S}(V, V_{\bar 0,\mathbb{R}})$.*

Definition C.36. Let E be a test space for $(V, V_{\bar 0,\mathbb{R}})$ and $\mu \in E'$, where we assume $\gamma_{\mathscr{S}}^\circ(\mu) \neq \varnothing$. The holomorphic superfunction on $T(\mu)$ defined in Theorem C.34 will be denoted by $\mathscr{L}(\mu)$ and called the *Laplace transform* of μ.

As above, the following is immediate from the classical case, which is [31, Proposition 8, Corollaire, Remarque].

Proposition C.37. *Let E be a test space for $(V, V_{\bar 0,\mathbb{R}})$ and $\mu \in E'$, where we assume $\gamma := \gamma_{\mathscr{S}}^\circ(\mu) \neq \varnothing$. Let $\xi \in V_{\bar 0,\mathbb{R}}^*$. Then the support of $\mu \in \mathscr{S}'(V, V_{\bar 0,\mathbb{R}})$ is contained in the half-space*

$$H_{\xi,C} := \big\{ v \in V_{\bar 0,\mathbb{R}} \;\big|\; \langle \xi, v\rangle \geqslant C \big\}$$

if and only if for every $c < C$ and for some, or equivalently, for all $\xi_0 \in \gamma_0$, we have $\xi_0 + \mathbb{R}_{\geqslant 0}\xi \subseteq \gamma_0$ and each of the functions

$$e^{tc}\mathscr{L}(\mu)(D;\xi_0 + t\xi + i\eta)$$

for $D \in S(V^*)$, is bounded for all $t \geqslant 0$ by some polynomial in η independent of t.

Corollary C.38. *Let $\mu \in \mathscr{S}'(V, V_{\bar{0},\mathbb{R}})$ and assume* $\operatorname{supp}\mu \subseteq \gamma$ *where the latter is a closed convex cone with $\gamma \cap (-\gamma) = 0$. Then*

$$\gamma^\circ_{\mathscr{S}}(\mu)_0 \supseteq \check{\gamma} := \big\{\xi \in V^*_{\bar{0},\mathbb{R}} \,\big|\, \forall v \in \gamma \setminus 0 : \langle \xi, v\rangle > 0\big\}.$$

Remark C.39. A nice and more elementary proof of the latter result (for the classical case) is given in Ref. [14].

References

[1] A. Alldridge, *The Harish-Chandra isomorphism for reductive symmetric superpairs*, Transformation Groups **17** (2012), no. 4, 889–919.

[2] A. Alldridge and J. Hilgert, *Invariant Berezin integration on homogeneous supermanifolds*, J. Lie Theory **20** (2010), no. 1, 65–91.

[3] G. E. Bredon, *Sheaf Theory*, second edition, Graduate Texts in Mathematics, vol. 170, Springer-Verlag, New York, 1997.

[4] P. Deligne and J.W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 1999, pp. 41–97.

[5] J. Dieudonné, *Treatise on Analysis. Vol. VI*, Academic Press, New York, 1978.

[6] L. Ehrenpreis, *Analytic functions and the Fourier transform of distributions. I*, Ann. of Math. (2) **63** (1956), 129–159.

[7] K. Efetov, *Supersymmetry in disorder and chaos*, Cambridge University Press, Cambridge, 1997.

[8] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), no. 1, 64–89, DOI 10.1016/0022-1236(90)90119-6.

[9] ———, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1994.

[10] Claudio Carmeli, Lauren Caston, and Rita Fioresi, *Mathematical Foundations of Supersymmetry*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2011.

[11] I. M. Gel'fand and G. E. Shilov, *Generalized Functions. Vol. 1*, Academic Press, New York, 1964.

[12] ———, *Generalized Functions. Vol. 2*, Academic Press, New York, 1977.

[13] V.W. Guillemin and S. Sternberg, *Supersymmetry and Equivariant De Rham Theory*, Springer-Verlag, Berlin, 1999.

[14] J. Hilgert and K.-H. Neeb, *Vector valued Riesz distributions on Euclidian Jordan algebras*, J. Geom. Anal. **11** (2001), no. 1, 43–75.

[15] A. E. Ingham, *An integral which occurs in statistics*, Proc. Cambridge Philos. Soc. **29** (1933), 271–276.

[16] T. Ishihara, *On generalized Laplace transforms*, Proc. Japan Acad. **37** (1961), 556–561.

[17] B. Iversen, *Cohomology of Sheaves*, Universitext, Springer-Verlag, Berlin, 1986.

[18] A. Kriegl and P. W. Michor, *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997.

[19] D. A. Leĭtes, *Introduction to the theory of supermanifolds*, Uspekhi Mat. Nauk **35** (1980), no. 1(211), 3–57, 255 (Russian).

[20] P. Littelmann, H.-J. Sommers, and M. R. Zirnbauer, *Superbosonization of invariant random matrix ensembles*, Comm. Math. Phys. **283** (2008), no. 2, 343–395.

[21] S. MacLane, *Categories for the Working Mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[22] Y. I. Manin, *Gauge Field Theory and Complex Geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 289, Springer-Verlag, Berlin, 1988. Translated from the Russian by N. Koblitz and J. R. King.

[23] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Oxford Graduate Texts in Mathematics, vol. 2, Oxford University Press, New York, 1997.

[24] G. Kainz, A. Kriegl, and P. Michor, *C^∞-algebras from the functional analytic viewpoint*, J. Pure Appl. Algebra **46** (1987), no. 1, 89–107.

[25] B. Mitiagin, S. Rolewicz, and W. Żelazko, *Entire functions in B_0-algebras*, Studia Math. **21** (1961/1962), 291–306.
[26] K.-H. Neeb, *Monastir Summer School: Infinite-dimensional Lie groups*, 2009. http://cel.archives-ouvertes.fr/docs/00/39/17/89/PDF/CoursKarl-HermannNeeb.pdf.
[27] P.-E. Paradan, *Symmetric spaces of the non-compact type: Lie groups*, Géométries à courbure négative ou nulle, groupes discrets et rigidités, Sémin. Congr., vol. 18, Soc. Math. France, Paris, 2009, pp. 39–76.
[28] M. Vergne and H. Rossi, *Analytic continuation of the holomorphic discrete series of a semisimple Lie group*, Acta Math. **136** (1976), no. 1-2, 1–59.
[29] M. Spivak, *A Comprehensive Introduction to Differential Geometry. Vol. I*, 2nd ed., Publish or Perish Inc., Wilmington, Del., 1979.
[30] H. H. Schaefer, *Topological vector spaces*, Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, 1971.
[31] L. Schwartz, *Transformation de Laplace des distributions*, Medd. Lunds Univ. Mat. Sem. **1952** (1952), no. Tome Supplementaire, 196–206.
[32] ———, *Théorie des distributions. Tome II*, Hermann, Paris, 1951.
[33] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. of Math. (2) **36** (1935), no. 3, 527–606, DOI 10.2307/1968644 (German).
[34] F. Trèves, *Topological Vector Spaces, Distributions, and Kernels*, Academic Press, New York, 1967.
[35] E. Witten, *Notes on supermanifolds and integration*, 2012. http://arxiv.org/abs/1209.2199.

Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln
E-mail address: `alldridg@math.uni-koeln.de`

Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln
E-mail address: `zain@math.uni-koeln.de`